A Counterexample to Tensorability of Effects

Sergey Goncharov and Lutz Schröder

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Forschungszentrum
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Very-very Abstract Picture

- Our work addresses the question of general existence of tensor products of monads open since 1969:

  Ernie Manes. A triple theoretic construction of compact algebras. In *Seminar on Triples and Categorical Homology Theory*, volume 80 of *Lect. Notes Math.*, pages 91–118. Springer, 1969.

- This work is a complementary part of our paper

  Sergey Goncharov and Lutz Schröder. Powermonads and tensors of unranked effects. In *LICS*, pages 227–236, 2011.

Monads, Effects and Metalanguage

Strong monad T: Underlying category $\mathcal{C}$, endofunctor $T : \mathcal{C} \to \mathcal{C}$, unit: $\eta : \mathrm{Id} \to T$, multiplication $\mu : T^2 \to T$,

(!) plus *strength*: $\tau_{A,B} : A \times TB \to T(A \times B)$.

Metalanguage of effects:

- $\mathsf{Type}_W ::= W \mid 1 \mid \mathsf{Type}_W \times \mathsf{Type}_W \mid T(\mathsf{Type}_W)$
- Term construction ((co-)Cartesian operators omitted):

$$\frac{x : A \in \Gamma}{\Gamma \rhd x : A} \qquad \frac{\Gamma \rhd t : A}{\Gamma \rhd f(t) : B}\ (f : A \to B \in \Sigma)$$

$$\frac{\Gamma \rhd t : A}{\Gamma \rhd \mathsf{ret}\, t : TA} \qquad \frac{\Gamma \rhd p : TA \quad \Gamma, x : A \rhd q : TB}{\Gamma \rhd \mathsf{do}\ x \leftarrow p; q : TB}$$

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Monads, Effects and Metalanguage: Usage

Rough idea:

- function spaces are morphisms: $[\![A \rightharpoonup B]\!] = [\![A]\!] \to T[\![B]\!]$;
- sequencing is binding: $[\![x := p; q]\!] = \mathrm{do}\ x \leftarrow [\![p]\!]; [\![q]\!]$;
- values are pure computations: $[\![\mathbf{c}]\!] = \mathrm{ret}[\![\mathbf{c}]\!]$.

Examples:

- Exceptions: $TA = A + E$.
- States: $TA = S \to (S \times A)$.
- Nondeterminism: $TA = \mathcal{P}(A)$, $\mathcal{P}_\omega(A)$, $\mathcal{P}^\star(A)$, ...
- Input/Output: $TA = \mu X.(A + (I \to O \times X))$.
- Continuations: $TA = (X \to R) \to R$.

For instance, for $TX = \mathcal{P}X$: $[\![A \rightharpoonup B]\!] = \mathcal{P}([\![A]\!] \times [\![B]\!])$.

Algebraic effects

(Finitary) Lawvere theory: small Cartesian category L plus a strict-product-preserving, identity-on-objects functor: $I : \mathbb{N}^{op} \to L$ ($\mathbb{N}$ = naturals and maps with summs as coproducts.)

- $L(n, 1)$ — operations; $L(0, 1)$ — constants;
- $\mathrm{Mod}(L, C) \subseteq \mathrm{Fun}(L, C)$ — models of L in C;
- forgetful functor $\mathrm{Mod}(L, C) \to C$ leads to finitary monads.

Finite nondeterminism: one constant $\bot : 0 \to 1$, one operation: $+ : 2 \to 1$. Then e.g. $(\lambda a, b, c.\, a + b + c) : 3 \to 1$, $(\lambda a.\, \langle a, \bot\rangle) : 1 \to 2$, etc.

States: $\mathtt{lookup}_l : V \to 1$, $\mathtt{update}_{l,v} 1 \to 1$ ($l \in L$, $v \in V$). E.g.: $\mathtt{update}_{l,v}(\mathtt{lookup}_l\langle p_1, \ldots, p_{|V|}\rangle) = \mathtt{update}_{l,v}(p_v)$.

Large Lawvere theory: L has all small products; $I : \mathbf{Set}^{op} \to L$ is strict-small-product-preserving, id-on-objects.

Theorem [Linton, 1966]: Large Lawvere theories = Monads on $\mathbf{Set}$.

Sum and Tensor

Sum of effects: blind union of signatures. For example
$\Sigma^\star + T = \mu\gamma.T(\Sigma\gamma + -)$ ($\Sigma^\star$ = I/O, Resumptions, Exeptions.)

Tensor = Sum modulo commutativity of operations:

$$\begin{array}{ccc} n_1 \times n_2 & \xrightarrow{n_1 \otimes f_2} & n_1 \times m_2 \\ {\scriptstyle f_1 \otimes n_2}\downarrow & & \downarrow{\scriptstyle f_1 \otimes m_2} \\ m_1 \times n_2 & \xrightarrow{m_1 \otimes f_2} & m_1 \times m_2. \end{array}$$

($n \otimes f = f \times \ldots \times f$ 'n times'.)

For instance: $\mathrm{lookup}_l\langle p_1 + q_1, p_2 + q_2\rangle$
$= \mathrm{lookup}_l\langle p_1, p_2\rangle + \mathrm{lookup}_l\langle q_1, q_2\rangle$.

Examples: $(- \times S)^S \otimes T = T(- \times S)^S$, $(-)^S \otimes T = T^S$,
$(M \times -) \otimes T = T(M \times -)$ where M is a monoid (of messages).

Tensors and Powermonads

Tensors can be used as monad transoformers. Example:

$$\mathsf{T} \otimes (S \times -)^S = \big(\mathsf{T}(S \times -)\big)^S$$

Another example: $\mathsf{T}^{\mathcal{P}} = \mathsf{T} \otimes \mathcal{P}$ — a powermonad. Provided existence of $\mathsf{T}^{\mathcal{P}}$,

- $\mathsf{T} \mapsto \mathsf{T}^{\mathcal{P}}$ is the left adjoint to the forgetful functor from completely additive monad (those enriched over complete semilattices with the bottom) to vanilla monads.
- $\mathsf{T}^{\mathcal{P}}$ supports generalised Fischer-Ladner encoding:

$$\begin{aligned} \mathit{if}(b, p, q) &:= \mathsf{do}\ b?; p + \mathsf{do}\,(\neg b)?; q, \\ \mathit{while}(b, p) &:= \mathsf{do}\ x \leftarrow (\mathsf{init}\, x \leftarrow \mathsf{ret}\, x\, \mathsf{in}(\mathsf{do}\ b?; p)^\star); \\ &\ \mathsf{do}\,(\neg b)?; \mathsf{ret}\, x \end{aligned}$$

Existence of tensors has been open since [Manes, 1969]

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Existence of Tensors

Existence of a tensor with $T \iff$ Smallness of $L_T(n, 1)$.

From **[Hyland, Plotkin, and Power, 2003]**, **[Hyland, Levy, Plotkin, and Power, 2007]** we know:

- tensors of ranked ($\approx$ algebraic) monads always exist;
- tensors of ranked monads with continuations exist;
- tensors with states always exist.

Tensors with uniform monads exist **[Goncharov and Schröder, 2011]** (e.g. $\mathcal{P}$ and the continuations are uniform).

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Example: $\mathcal{P} \otimes T$ exists for $L_{\mathcal{P}\otimes T}(n, 1)$ is a quotient of $\mathcal{P}(L_T(n, 1))$. For instance if $TX = \mu\gamma.\,(\gamma \times \gamma + X)$ then

$$f(\{a, b\}, c) \to f(\{a\} \cup \{b\}, \{c\} \cup \varnothing) \to \{f(a, c), f(b, \varnothing)\}$$

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Non-existence of Tensors: Plan of the Proof

1. Define an unranked non-uniform monad $\mathcal{W}$.
   Let $\mathcal{W} + 2 = \mathcal{W}(- + 2)$.
2. For every S and T introduce $(S \otimes T)$-algebras, which are simultaneously S- and T-algebras satisfying commutation of S-operations with T-operations.
3. Ensure that whenever $(S \otimes T)\varnothing$ exists it must be the initial $(S \otimes T)$-algebra.
4. Find such T for which there are $((\mathcal{W} + 2) \otimes T)$-algebras of arbitrary large cardinality with "no junk".
5. ?????
6. PROFIT!!!

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Tensor Algebras

Definition: Given two monads T and $\mathcal{S}$, $(T \otimes \mathcal{S})$-*algebras* are triples of the form (X, α, β) where (X, α) is a T-algebra, (X, β) is an $\mathcal{S}$-algebra, and moreover for all sets Y, Z and all $p \in \mathcal{S}Y$, $q \in TZ$, $f : Y \times Z \to X$,

$$\beta(T(\lambda z.\, \alpha(\mathcal{S} f_{-,z}\ p))q) = \alpha(\mathcal{S}(\lambda y.\, \beta(T f_{y,-}\ q))p)$$

where $f_{-,z}(y) = f_{y,-}(z) = f(y, z)$ for $(y, z) \in Y \times Z$.

Theorem: The tensor $T \otimes \mathcal{S}$ of monads $T, \mathcal{S}$ exists iff the forgetful functor from $(T \otimes \mathcal{S})$-algebras to $\mathbf{Set}$ is monadic, equivalently has a left adjoint.

Corolary: If the tensor $T \otimes \mathcal{S}$ of monads T and $\mathcal{S}$ exists, then there exists an initial $(T \otimes \mathcal{S})$-algebra.

The Well-order Monad

Definition: A $\mathcal{W}$-*algebra* is a set X equipped with an ordinal-indexed family of operations $\iota_\kappa : X^\kappa \to X$ satisfying the conditions:

1. *strictness:* $\iota_\kappa(w) = \iota_0$ if $w(\alpha) = \iota_0$ for some $\alpha < \kappa$.
2. *non-repetitiveness:* $\iota_\kappa(w) = \iota_0$ whenever $w(\alpha_1) = w(\alpha_2)$ for some $\alpha_1 < \alpha_2 < \kappa$.
3. *associativity:* for every ordinal-indexed family $(\kappa_\mu)_{\mu<\nu}$ of ordinals $\kappa_\mu > 0$, $\iota_\kappa(w) = \iota_\nu(\lambda\mu < \nu.\, \iota_{\kappa_\mu}(w_\mu))$.

Theorem: $\mathcal{W}$-algebras give rise to a monad $\mathcal{W}$. Specifically,

$$\mathcal{W}X = \{(Y, \rho) \mid Y \subseteq X, \rho \text{ a well-order on } Y\}.$$

Equivalently, $\mathcal{W}X$ can be considered as the set of all non-repetitive ordinal-indexed lists.

The Counterexample

Let $\Sigma_{2,2} = \lambda X.\, 2 \times X \times X$. Then $(\mathcal{W}(-+2) \otimes \Sigma^{\star}_{2,2})$ does not exist since for every κ there is a reachable $(\mathcal{W}(-+2) \otimes \Sigma^{\star}_{2,2})$-algebra W_κ.

The domain of W_κ consists of terms involving

- constants 0, 1, $\perp$,
- binary operations u_0, u_1,
- ordinal-indexed of κ-bounded lists.

formed by the rules

$$\frac{t \in W_\kappa - \{0\}}{u_0(0,t) \in U^0_\kappa} \qquad \frac{t \in W_\kappa - \{0\}}{u_1(0,t) \in U^1_\kappa}$$

$$\frac{1 < |\nu| \leqslant \kappa \qquad t : \nu \hookrightarrow U^0_\kappa \cup U^1_\kappa \qquad \forall \mu.\ \mu + 1 < \nu \implies \left(t(\mu) \in U^0_\kappa \iff t(\mu+1) \in U^1_\kappa\right)}{t \in L_\kappa}$$

where $W_\kappa = \{\perp, 0, 1\} \cup U^0_\kappa \cup U^1_\kappa \cup L_\kappa$.

Tensoring with Finite Lists

- Let L be the large Lawvere theory for non-empty lists.
- Observe that L is generated by one binary operation u and the associativity axiom:

$$u(u(a,b),c) = u(a,u(b,c)).$$

- Now the tensor $\mathrm{L} \otimes \mathrm{L}$ is obtained from $\Sigma^{\star}_{2,2} = \Sigma^{\star}_{2,1} + \Sigma^{\star}_{2,1}$ by quotiening under the associativity and the tensor laws (u' is a duplicate of u):

$$u'(u(a_1,b_1),u(a_2,b_2)) = u(u'(a_1,a_2),u'(b_1,b_2)).$$

- Observe that W_κ is a $(\mathrm{L} \otimes \mathrm{L}) \otimes (\mathcal{W} + 2)$-algebra. Hence $\mathrm{L} \otimes (\mathrm{L} \otimes (\mathcal{W} + 2)) = (\mathrm{L} \otimes \mathrm{L}) \otimes (\mathcal{W} + 2)$ does not exist.

Theorem: The non-empty list monad is not tensorable.

Conclusions

- Unranked monads are often tensorable.
  - Including continuations.
- We have provided a counterexample for tensorability of effects
  - We have introduced the well-order monad.
  - We have shown that the tensor with a simple ranked monad need not exist.
  - We have shown that the tensor with the list monad need not exist.

Future Work

- Find more applications of the tensor product.
  - Give a monad-based account of separation logic.
- Extends the existence result to capture more partial cases uniformly.

The End

Thanks for your attention!

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