Game Harmony: A Short Note

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August 2000

Strategic uncertainty in game theory may have two di¤erent general sources, either alone or in combination: uncertainty because of the existence of a coordination problem, and uncertainty because of a con‡ict between one own and the other n players' interests. Game harmony is conceived as a generic game property that describes how harmonious (non-con‡ictual) or disharmonious (con‡ictual) the interests of the n players are, as embodied in the game payo¤s. Pure coordination games are examples of games with maximal game harmony; zero sum games are examples of games with very low game harmony.

This note brie‡y describes attempts to measure game harmony simply as a real-valued number. These attempts do not focus just on equilibrium payo¤s. This is not just because this would make the measurement of game harmony dependent on the reasonableness of the equilibrium concept for any given game, but also and more fundamentally because a restriction on equilibrium payo¤s would hide any disharmony in the interests among players that does not derive from the equilibrium payo¤s. For example, in the case of the standard Prisoner's Dilemma, the equilibrium payo¤ gives the same amount to each player, and so no disharmony of interests emerges while it should.

A simple way to think about game harmony measurement would be to ask ourselves to what extent the payo¤s of each player co-vary. In the case of a pure coordination game, they co-vary one-to-one and positively, whereas in the case of a zero-sum game they co-vary one-to-one but negatively. Therefore, parametric or nonparametric correlation measures, such as Pearson or Spearman respectively, may be used as game harmony measures. Unfortunately, these measures can only be used with $n = 2$. Some form of extension is required if there are more than two players. Let k be the number of binary combinations among players:

(1) $k = \frac{n!}{r!(n-r)!}$

Let r_i and ρ_i be the Pearson and Spearman correlations, respectively, in relation to any such combination, $\forall i \in [1;k]$. We can then de…ne two game harmony indexes, GH_P and GH_S, as:

(2) $GH_P = \frac{\sum_{i=1}^{k} r_i}{n}$

(3) $GH_S = \frac{\sum_{i=1}^{k} \rho_i}{n}$

(2) and (3) are bounded between -1 and 1, and a higher index value corresponds to higher game harmony. If $n = 2$, then $k = 1$, so the indexes collapse to the simple value of the Pearson or the Spearman correlation.

One might think at more "self-centred" psychological measures of game harmony, from the perspective of any individual player i: one could use a weighted average of r_i giving more weight to the combinations of which player s is a member. There will be $n - 1$ such combinations. De…ne r_{sj} as the Pearson correlation of player s' payo¤s with each of the other $(n - 1)$ players. Then, in the polar case in which weight is exclusively put on one own combinations, the i-player's "self-centred" game harmony would be formulated as:

(4) $GH_P^s = \frac{\sum_{j=1}^{n-1} r_{sj}}{n-1}$

and, proceeding similarly, one could de…ne a GH_S^s index based on the Spearman measure of correlation.

An alternative class of game harmony measures is based on traditional measures of income distribution, and can be labeled IDM measures. Measures of income distribution can be useful to summarise information on the payo¤ distribution for each payo¤ outcome, with no general restriction on the number of players. Let a_{sj} be the payo¤ of player S for some given game outcome j (out of m possible outcomes). IDM measures can be computed in three steps.

The …rst step is ratio-normalisation of payo¤s, by …nding, for each payo¤ value,

(5) $a^{*}_{sj} = \frac{a_{sj}}{\sum a_{sj}}$

i.e., by dividing each payo¤ by the sum of all possible payo¤s for the player. Ratio-normalisation is essential because, otherwise, the IDM measure would mirror the average payo¤ distribution of the game, but not the extent to which the players' interests in achieving one or another game outcome are the same or are in con‡ict. As an example, take the following simple 2x2 game:

	Left	Right
Top	4, 2	1, 0.5
Bottom	1, 0.5	2, 1

This is a game of perfect harmony, in the sense that the interest of each player of achieving any outcome is identical, but an IDM measure would not be able to capture this unless ratio-normalisation (or a similar algorithm) is used.

The second step is to compute the payoff distribution index for each possible outcome j, using the algorithm of one of the standard indexes of income distribution, such as Gini (1910) or Theil (1967). A brief discussion of what indexes may be more suitable is contained below. Generally speaking, for every outcome j, we find a real number I_j that is a function of the payoffs of all possible players given this outcome. More formally,

(6) $I_j = i(a^*_{1j}, a^*_{2j}, ..., a^*_{nj})$

We can then define the IDM game harmony index as:

(7) $GH_{IDM} = \frac{\sum_{j=1}^{m} I_j}{m}$

One general feature of IDM game harmony indices is that a greater value corresponds to less, and not more, game harmony, unlike the correlation-based indices. Of course, one could easily avoid this source of confusion by, say, reparametrising (7) in terms of $(1 - GH_{IDM})$ rather than dealing directly with GH_{IDM}.

What index of income distribution might be potentially good candidates to measure game harmony? One restriction is imposed by the constraints that many games of economic interest have few players, even just two players: the index must be defined and retain desirable features for $n = 2$. For example, the relative mean deviation criterion (Cowell, 1978) is not applicable in this case; while the value of the Gini index is rather sensitive to the value of n for low n values, and has a maximum value of 1/2 rather than 1 for $n = 2$. To avoid the latter problem, one can normalise the Gini index by multiplying its value by $n/(n-1)$: labeling this normalised index as I_j^G for some outcome j, and ordering subjects from "poorest" (i.e., that with the lowset payoff) to "wealthiest" (i.e., that with the highest payoff), we can define I_j^G as:

(8) $I_j^G = \frac{n}{n-1}\frac{2}{n\sum_{s=1}^{n} a^*_{sj}} \sum_{s=1}^{n} s\left(a^*_{sj} - \frac{\sum_{s=1}^{n} a^*_{sj}}{n}\right) =$

$= \frac{1}{n-1}\frac{2}{\sum_{s=1}^{n} a^*_{sj}} \sum_{s=1}^{n} s\left(a^*_{sj} - \frac{\sum_{s=1}^{n} a^*_{sj}}{n}\right)$

One can then find the normalised Gini-based game harmony index GH_G:

(9) $GH_G = \frac{\sum_{j=1}^{m} I_j^G}{m}$

With $n = 2$, I_j^G simplifies to the following:

(10) $I_j^G = \frac{2}{a^*_{1j}+a^*_{2j}}\left[\left(a^*_{1j} - \frac{a^*_{1j}+a^*_{2j}}{2}\right) + 2\left(a^*_{2j} - \frac{a^*_{1j}+a^*_{2j}}{2}\right)\right] =$

$$= \frac{2}{a^*_{1j}+a^*_{2j}} \left(\frac{2a^*_{1j} - a^*_{1j} - a^*_{2j} + 4a^*_{2j} - 2a^*_{1j} - 2a^*_{2j}}{2} \right) = \frac{a^*_{2j}-a^*_{1j}}{a^*_{1j}+a^*_{2j}}$$

One possible criticism of the normalised Gini criterion is that it loses its boundedness between 0 and 1 if there are negative payoffs. This is, of course, a general problem of the Gini index, but it may be more serious in dealing with game payoffs than it is when thinking about incomes. Ways to further normalise the Gini index to preserve the 0, 1 boundary however exist (Chen et al., 1982; Van de Ven, 2000).

The Theil and the Herfindahl indexes are other possible candidates for I_j: Instead income distribution criteria that overweigh low relative to high payoffs (such as logarithmic variance: Cowell, 1978) do not appear useful, since it is not clear how the overweighting may be justified, in relation to game harmony. Criteria stating what fraction of income/payoff is below some target level (e.g., the minimal majority criterion) also do not appear to have any obvious justification in relation to game harmony. A similar point can be made in relation to criteria of income distribution based on social welfare functions.

References

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