The intensity JND comes from Poisson neural noise: Implications for image coding

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ABSTRACT

While the problems of image coding and audio coding have frequently been assumed to have similarities, specific sets of relationships have remained vague. One area where there should be a meaningful comparison is with central masking noise estimates, which define the codec's quantizer step size. In the past few years, progress has been made on this problem in the auditory domain (Allen and Neely, J. Acoust. Soc. Am., **102**, 1997, 3628-46; Allen, 1999, Wiley Encyclopedia of Electrical and Electronics Engineering, Vol. 17, p. 422-437, Ed. Webster, J.G., John Wiley & Sons, Inc, NY). It is possible that some useful insights might now be obtained by comparing the auditory and visual cases. In the auditory case it has been shown, directly from psychophysical data, that below about 5 sones (a measure of loudness, a unit of psychological intensity), the loudness JND is proportional to the square root of the loudness $\Delta\mathcal{L}(\mathcal{L}) \propto \sqrt{\mathcal{L}(I)}$. This is true for both wideband noise and tones, having a frequency of 250 Hz or greater. Allen and Neely interpret this to mean that the internal noise is Poisson, as would be expected from neural point process noise. It follows directly that the Ekman fraction (the relative loudness JND), decreases as one over the square root of the loudness, namely $\Delta\mathcal{L}/\mathcal{L} \propto 1/\sqrt{\mathcal{L}}$. Above $\mathcal{L} = 5$ sones, the relative loudness JND $\Delta\mathcal{L}/\mathcal{L} \approx 0.03$ (i.e., Ekman law). It would be very interesting to know if this same relationship holds for the visual case between brightness $\mathcal{B}(I)$ and the brightness JND $\Delta\mathcal{B}(I)$. This might be tested by measuring both the brightness JND and the brightness as a function of intensity, and transforming the intensity JND into a brightness JND, namely

$$\Delta\mathcal{B}(I) = \mathcal{B}(I + \Delta I) - \mathcal{B}(I) \approx \Delta I \frac{d\mathcal{B}}{dI}.$$

If the Poisson nature of the loudness relation (below 5 sones) is a general result of central neural noise, as is anticipated, then one would expect that it would also hold in vision, namely that $\Delta\mathcal{B}(\mathcal{B}) \propto \sqrt{\mathcal{B}(I)}$. It is well documented that the exponent in the S.S. Stevens' power law is the same for loudness and brightness (Stevens, 1961) (i.e., both brightness $\mathcal{B}(I)$ and loudness $\mathcal{L}(I)$ are proportional to $I^{0.3}$). Furthermore, the brightness JND data are more like Riesz's near miss data than recent 2AFC studies of JND measures.[2,3]

Key Words: Contrast JND, Poisson noise, Brightness

1. INTRODUCTION

When modeling human psychophysics one must carefully distinguish the external *physical* variables, which we call Φ variables, from the internal (i.e., loudness and brightness) *psychophysical* variables, which we refer to as Ψ variables. Psychophysical modeling seeks a transformation from the Φ domain to the Ψ domain. The Φ–intensity is easily quantified by direct measurement. The auditory Ψ–intensity is the *loudness*, while in vision it is called the *brightness*. The idea that the Ψ–intensity could be quantified was suggested by Fechner (1966) in 1860, who was first to raise the possibility of defining a quantitative transformation between the physical and psychophysical intensity.[4]

An increment in the intensity of a sound that results in the *just noticeable difference* is called an intensity JND, or simply the *difference limen* (DL). Fechner suggested quantifying the $\Psi(\Phi)$ transformation by counting the number of $\Psi(I)$–JNDs between any two intensity values. However, after many years of work, this relationship (e.g., between loudness and the intensity JNDs) has remained unclear.[5–7,3]

Since the classic 1927 work of Thurstone,[8] it has been widely accepted that the intensity JND is the physical correlate of the *psychological domain uncertainty* (internal noise) corresponding to the *psychological intensity*. For example, the loudness JND is a measure of loudness noise. To model the intensity JND one must define a *decision variable*, associated with random Ψ–domain fluctuations.In the auditory case, this random variable has been called the *single–trial loudness* $\widetilde{\mathcal{L}}(I)$.[9] The loudness $\mathcal{L}(I)$ and the loudness JND $\Delta\mathcal{L}(I)$ are defined in terms of the first and second moments of the single–trial loudness, corresponding to the mean and variance of the distribution of the intensity decision variable. Given $\mathcal{L}(I)$ and $\Delta I(I)$, one may transform $\Delta I(I)$ into $\Delta\mathcal{L}(\mathcal{L}) \equiv \mathcal{L}(I + \Delta I) - \mathcal{L}(I)$. The ratio of the loudness over the loudness standard deviation is defined as the *loudness signal–to–noise ratio* $\mathrm{SNR}_{\mathcal{L}} \propto \mathcal{L}/\sigma_{\mathcal{L}}$, which is proportional to the reciprocal of the Ekman fraction for loudness $\Delta\mathcal{L}(\mathcal{L})/\mathcal{L}$.

Allen and Neely (1997) found the same functional dependence of $\Delta\mathcal{L}(\mathcal{L})$ for both tones and wideband noise. This is surprising since this unified Riesz's tonal DL "near–miss to Weber's law" data and Miller's wideband noise DL data, which satisfies Weber's law. Furthermore,

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Allen and Neely provided a simple physical explanation of the resulting $\Delta\mathcal{L}(\mathcal{L})$ response. This work unifies masking and the JND, following the 1947 outline of this problem by Miller (1947). The Allen and Neely approach is conceptually similar to that of Baird and Noma (1978), page 84–85 and Ekman.[12] It is expected, based on simple common-sense physical arguments, that brightness must follow a similar relationship for intensities, when neural noise dominates the brightness JND.

For the case of tones, Allen and Neely used the intensity modulation results of Riesz (1928) with the loudness data of Fletcher and Munson (1933) to do these calculations. Riesz measured the intensity JND using a pair of tones (one large and small) having a frequency difference of 3 Hz. "Modern" methods generally use "pulsed" tones, which are turned on and off somewhat abruptly, making them unsuitable for comparison to the 1 second loudness measurements of Fletcher and Munson. Riesz's modulation method has a distinct advantage in characterizing the internal signal detection process because it maintains a tone-like, threshold modulation condition. The interpretation of Riesz's intensity JNDs is therefore simplified since all underlying stochastic processes are stationary (i.e., the stimuli for the JND measurements are close to pure tones, and have a long duration, like the loudness data).

Since Allen and Neely (1997) use the Fletcher and Munson's 1933 loudness data, the results are expected to be more accurate than methods based on Stevens scaling, as explained by Fletcher and Munson (1933). This may be a factor in the Allen and Neely estimate of $\Delta\mathcal{L}(\mathcal{L})$.

2. BASIC DEFINITIONS

Intensity. The Φ *intensity* is a power per unit area. In the time domain, it is common to define the Φ–*intensity* in terms of the time-integrated squared signal pressure $s(t)$, namely,*

$$I_s(t) \equiv \frac{1}{\varrho c T}\int_{t-T}^{t} s^2(t)dt, \tag{1}$$

where T is the integration time and ϱc is the specific acoustic impedance of air.

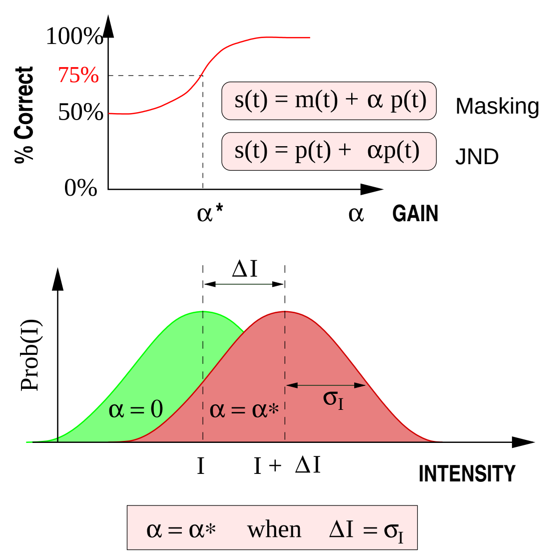

Figure 1. *Following Signal Detection Theory, we assume that when two intervals are used, 75% of the time the observer can correctly identify the interval with the probe. For one half of the presentations the probe is heard, and therefore scored correctly. For the other half of the presentations, when it is not heard, the subject guesses correctly 1/2 the time, resulting in a 50% + 25% = 75% correct total score.*

Intensity of masker + probe. The JND is sometimes described as "self-masking," to reflect the view that it is determined by the internal noise of the auditory system. To model the JND it is useful to define a more general measure called the *masked threshold*, which is defined in the Φ domain in terms of a pressure scale factor α applied to the probe signal $p(t)$ that is then added to the masking pressure signal $m(t)$. The relative intensity of the probe and masker is varied by changing α. Setting $s(t) = m(t) + \alpha p(t)$, we denote the combined intensity as

$$I_{m+p}(t,\alpha) \equiv \frac{1}{\varrho c T}\int_{t-T}^{t} (m(t) + \alpha\, p(t))^2 dt. \tag{2}$$

The unscaled probe signal $p(t)$ is chosen to have the same long–term average intensity as the masker $m(t)$, defined as I. Let $I_m(t) = I$ be the intensity of the masker with no probe ($\alpha = 0$), and $I_p(t,\alpha) = \alpha^2 I$ be the intensity of the scaled probe signal with no masker.

Beats. Rapid fluctuations having frequency components outside the bandwidth of the T duration rectangular integration window are very small and will be ignored. Accordingly we drop the time dependence in terms I_m and I_p. Because of beats between $m(t)$ and $p(t)$ (assuming

*The symbol $\equiv$ denotes "equivalence." It means that the quantity to the left of the $\equiv$ is defined by the quantity on the right.

the spectra of these signals are within a common critical band) one must proceed carefully. Slowly varying correlations between the probe and masker having frequency components within the bandwidth of the integration window may *not* be ignored, as with beats between two tones separated in frequency by a few Hz. Accordingly we keep the time dependence in the term $I_{m+p}(t, \alpha)$ and other slow–beating time dependent terms. In the Φ domain these beats are accounted for with a probe–masker correlation function $\rho_{mt}(t)$.[15,16]

Intensity increment $\delta I(t, \alpha, I)$. Expanding Eq. 2 and solving for the *intensity increment* $\delta I(t, \alpha, I)$ we find

$$\delta I(t, \alpha, I) \equiv I_{m+p}(t, \alpha) - I \tag{3}$$

$$= \left(2\alpha\rho_{mp}(t) + \alpha^2\right) I, \tag{4}$$

where

$$\rho_{mp}(t) = \frac{1}{\varrho c T I} \int_{t-T}^{t} m(t)p(t)dt \tag{5}$$

defines a normalized cross correlation function between the masker and the probe. The correlation function must lie between -1 and 1.

The detection threshold. As shown in Fig. 1, when the probe to masker ratio α is slowly increased from zero, the probe can eventually be detected. We specify the *detection threshold* as α_*, where the asterisk indicates the threshold value of α where a subject can discriminate intensity $I_{m+p}(t, \alpha_*)$ from intensity $I_{m+p}(t, 0)$ 50% of the time, corrected for chance [i.e., obtain a 75% correct score in a direct comparison of the two signals[17,16]]. The quantity $\alpha_*(t, I)$ is the probe to masker RMS pressure ratio at the detection threshold. It is a function of the masker intensity I and, depending on the experimental setup, time.

Masked threshold intensity. The *masked threshold intensity* is defined in terms of α_* as $I_p^*(I) \equiv I_p(\alpha_*) = \alpha_*^2 I$, which is the threshold intensity of the probe in the presence of the masker.

The masked threshold intensity is a strong function of the stimulus modulation parameters. For example, tone maskers and narrow–band noise maskers of equal intensity, and therefore approximately equal loudness, give masked thresholds that are about 20 dB different.[18] As a second example, when using the method of beats, the just–detectable modulation depends on the beat frequency.[13] With "modern" 2AFC methods , the signals are usually gated on and off (100% modulation).[19] According to Stevens and Davis (p. 142, 1983)

> A gradual transition, such as the sinusoidal variation used by Riesz, is less easy to detect than an abrupt transition; but, as already suggested, an abrupt transition may involve the production of unwanted transients.†

One must conclude, as is widely acknowledged, that the *relative masked threshold* [i.e., $\alpha_*(t, I)$] is a strong function of the modulation conditions. This dependence is due, in part, to the modulation filtering that takes place after the signal has been detected, which is sometimes called *temporal integration*. This is analogous to the spatial filtering of the eye.

Ψ–domain temporal resolution. The Ψ–domain temporal resolution plays a key role in intensity JND and masking models and the relevant integration time T is determined in the Ψ–domain. This important Ψ–domain model parameter is called *loudness temporal integration time*.[17] As far as I can determine, temporal integration was first explicitly modeled by Munson (1947). A closely related measure is the *modulation transfer function*, which is the frequency response of the loudness temporal integration filter to sinusoidal modulations.

The Φ–domain temporal resolution (T) and modulation transfer function are critical to the definition of the JND in Riesz's experiment [See the Appendix A of Allen and Neely (1997), as well as Riesz (1928)] because it determines the optimum threshold intensity of the beats. Beats cannot be heard if they are faster than, and therefore "filtered" out by, the Ψ domain response, or if they are too slow. Riesz found 3 Hz to be the optimum modulation frequency. Even though Riesz's modulation detection experiment is technically a masking task, we treat it, following Riesz (1928), Fletcher (1995), Miller (1947), and Littler (1965), as characterizing the intensity JND.

The Ψ–domain temporal resolution also impacts results for gated stimuli, such as in the 2AFC experiment, though its role is poorly understood in this case. More important, matching loudness measurements have not been made for gated stimuli, making direct JND and loudness comparisons impossible. For this reason we have restricted our analysis to the pure tone case (Riesz JNDs versus Fletcher–Munson loudness).

The intensity JND ΔI. The intensity *just-noticeable difference* (JND) is defined as‡

$$\Delta I(I) \equiv \delta(\alpha_*, I), \tag{6}$$

the intensity increment at the masked threshold. From Eq. 4, with $\alpha = \alpha_*$ and $\rho_{mp}(t) = 1$,

$$\Delta I(I) = (2\alpha_* + \alpha_*^2) I. \tag{7}$$

An important alternative definition for the special case of the *pure–tone JND* is to let the masker be a pure tone, and let the probe be a pure tone of a slightly different frequency (e.g., a beat frequency difference of f_b = 3 Hz). This was the definition used by Riesz in 1928. Beats are heard at f_b = 3 Hz, and $\rho_{mp}(t) = \sin(2\pi f_b t)$. Thus

$$\delta[t, \alpha_*, I] = [2\alpha_* \sin(2\pi f_b t) + \alpha_*^2] I, \tag{8}$$

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†I take this quote to mean a *transient in the envelope*, *not* a transient that produces out-of-band spectral splatter, which is commonly and easily controlled by ramping up and down the stimulus.

‡It is traditional to define the intensity JND to be a function of I, rather than a function of $\alpha(I)$, as we have done here. We shall treat both notations as equivalent [i.e., $\Delta I(I)$ or $\Delta I(\alpha(I))$].

and

$$\Delta I(I) \equiv \max_t \ \delta(t, \alpha_*, I) - \min_t \ \delta(t, \alpha_*, I), \tag{9}$$

which means $\Delta I(I) \approx 4\alpha_* I$ to a very good approximation (due to the small value[13] of $\alpha_* \approx 0.05$).

Internal noise. It is widely accepted that the pure–tone intensity JND is determined by the *internal noise* of the auditory system,[23,24] and that ΔI is proportional to the standard deviation of the Ψ–domain decision variable that is being discriminated in the intensity detection task, reflected back into the Φ domain. The usual assumption, from signal detection theory, is that $\Delta I = d'\sigma_I$, where $d' \equiv \Delta I/\sigma_I$ is a constant that depends on the experimental design, and σ_I is the intensity standard deviation of the Φ–domain intensity due to Ψ–domain auditory noise.[9,17]

Hearing threshold. The *hearing threshold* (or unmasked threshold) *intensity* may be defined as the intensity corresponding to the first (lowest) intensity JND. It is frequently used as the reference when expressing the masked threshold in dB. The hearing threshold is represented as $I_p^*(0)$ to indicate the probe intensity when the masker intensity is small (i.e., $I \to 0$). While it is believed that internal noise is responsible for the hearing threshold, there is no reason to assume that this noise is the same as the internal noise that produces the super-threshold JND.

Loudness $\mathcal{L}$. *Loudness*, in *sones* or *loudness units* (LU),[§] is the name commonly given to the Ψ intensity. The *loudness growth function* $\mathcal{L}(I)$ depends on the stimulus conditions. For example $\mathcal{L}(I)$ for a tone and for wideband noise are not the same functions. Likewise the loudness growth function for a 100 ms tone and a 1-s tone differ. When defining a *loudness scale* it is traditional to specify the intensity, frequency, and duration of a tone such that the loudness growth function is one (i.e., $\mathcal{L}(I_{\text{ref}}, f_{\text{ref}}, T_{\text{ref}}) = 1$ defines a loudness scale). For the sone scale, the reference signal is a I_{ref} = 40 dB SPL tone at f_{ref} = 1 kHz with duration T_{ref} = 1 s. For Fletcher's LU scale the reference intensity is the hearing threshold at 1 kHz, which means that 1 sone = 975 LU[25] for a "normal" hearing person.

Detection theory and the single-trial loudness. A fundamental postulate of modern psychophysics is that all perceptual (i.e., Ψ) variables are *random variables*.[8] For alternative discussions of this point see Montgomery (1935), p. 144 of Stevens and Davis (1983), and chapter 5 of Green (1966). To clearly indicate the distinction between random and nonrandom variables, a tilde overbar ($\sim$) is used to indicate every random variable.[¶]

We define the loudness decision variable as the *single–trial loudness* $\widetilde{\mathcal{L}}(I)$, which is the sample–loudness heard on each stimulus presentation. The loudness $\mathcal{L}$ is then the expected value of the single–trial loudness $\widetilde{\mathcal{L}}$

$$\mathcal{L}(I) \equiv \mathcal{E}\widetilde{\mathcal{L}}(I). \tag{10}$$

The second moment of the single-trial loudness

$$\sigma_{\mathcal{L}}^2 \equiv \mathcal{E}(\widetilde{\mathcal{L}} - L)^2 \tag{11}$$

defines the loudness *variance* $\sigma_{\mathcal{L}}^2$ and *standard deviation* $\sigma_{\mathcal{L}} = \Delta\mathcal{L}/d'$.

Loudness growth. Loudness depends in a complex manner on a number of acoustical variables, such as intensity, frequency, and spectral bandwidth, and on the temporal properties of the stimulus, as well as on the mode of listening (in quiet or in noise, binaural or monaural stimulation). Isoloudness contours describe the relation of equal loudness between tones or between narrow bands of noise at different frequencies.

In 1924 Fletcher and Steinberg published an important article on the measurement of the loudness of speech signals.[28] In this paper, when describing the growth of loudness, the authors state

> the use of the above formula involved a *summation of the cube root of the energy rather than the energy*.

This cube-root dependence was first described by Fletcher the year before.[29] Today any power-law relation between the intensity of the physical stimulus and the psychophysical response is referred to as *Stevens's law*.[30,17] Fletcher's 1923 loudness growth equation established the important special case of loudness for Stevens's approximate, but more general, psychological "law."

Cochlear compression. What is the source of Fletcher's cube root loudness growth (i.e., Stevens's law)? Today we know that the basilar membrane motion is nonlinear, and that basilar membrane stiffness changes, due to outer hair cells (OHC), are the source of the basilar membrane nonlinearity and the cube root loudness growth first observed by Fletcher.

From noise trauma experiments on animals and humans, it is now widely accepted that recruitment (abnormal loudness growth) occurs in the cochlea.[31] In 1937 Lorente de No theorized that recruitment is due to hair cell damage.[32] Animal experiments have confirmed this prediction and have emphasized the importance of outer hair cells (OHC) loss.[33,34] This loss of OHCs causes a modification of basilar membrane compression, first described by Rhode in 1971.[17,35]

We still do not know precisely what controls the basilar membrane nonlinearity (i.e., the exponent of the power law), although we know that it is related to outer hair cell stiffness changes[36,37] which are controlled by the OHC membrane voltage.[38] This voltage is determined by shearing displacement of the hair cell cilia by the tectorial membrane. We know that the inner hair cell (IHC) has a limited dynamic range of less than 60 dB, yet it is experimentally observed that these cells code a dynamic range of about 120 dB.[39] Nonlinear compression by cochlear OHCs, prior to IHC detection, increases the dynamic range of the IHC detectors. When the OHCs are damaged, the compression becomes linear, and loudness recruitment results.[40]

[§] Sones and LU are related by a scale factor: 1 Sone is 975 LU.

[¶] As a mnemonic, think of the $\sim$ as a "wiggle" associated with randomness.

Loudness additivity. Fletcher and Munson (1933) showed, for tonal stimuli, *(1)* the relation of iso-loudness across frequency (loudness-level in phons), *(2)* the dependence of loudness on intensity *(3)* a model showing the relation of masking to loudness, and *(4)* the basic idea behind the critical band (critical ratio). Possibly even more important, they were the first to introduce a totally new concept, *loudness additivity*.[41] They presented a huge amount of empirical data to support this radical idea.

Rather than thinking directly in terms of loudness growth, they tried to find a formula describing how the loudnesses of several stimuli combine. From loudness experiments with low- and highpass speech and complex tones[28,42] and from other unpublished experiments over the previous 10 years, they found that loudness adds. Their hypothesis was that when two equally loud tones that do not mask each other are presented together, the result is "twice as loud." They showed that when N tones that are all equally loud are played together, the result is N times louder (for N up to 11), as long as they do not mask each other. Fletcher and Munson found that loudness additivity holds for signals between the two ears as well as for signals in the same ear. When the tones masked each other (namely, when their masking patterns overlapped), additivity still holds, but over an attenuated set of patterns,[41] since the overlap region must not be counted twice. Their 1933 model is fundamental to our present understanding of auditory sound processing.

The argument. Let $G(p_1, p_2)$ be the nonlinear compression function that maps the ear canal pressure p_1 at frequency f_1 and p_2 at f_2 into the loudness in sones, *under the condition that the tones are far enough apart in frequency that they do not mask each other*. When one tone masks another, the loudness $\mathcal{L}$ is always less than G (i.e., masking always reduces the loudness). When each tone is presented alone, there is no masking, so $\mathcal{L} = G$. It also follows that $\mathcal{L}_1 = G(p_1, 0)$ and $\mathcal{L}_2 = G(0, p_2)$. We assume that $G(0,0) = 0$ and $G(p_{\text{ref}}, 0) = 1$, where p_{ref} is either 20 μPa or the threshold of hearing at 1 kHz. The problem is to find $G(p_1, p_2)$.

Step 1. The pressure p_1 is taken as the reference level for the experiment with $f_1 = 1$ kHz. The level of pressure p_2, at frequency f_2, is next determined by requiring that its loudness be equal to that of p_1. We call this pressure $p_2^*(p_1, f_2)$, since it is a function of both p_1 and f_2. In terms of the compression function G, p_2^* is defined by

$$G(0, p_2^*) = G(p_1, 0). \qquad (12)$$

Step 2. Fletcher and Munson scaled the reference pressure p_1 by scale factor α and defined α^* such that the loudness of $\alpha^* p_1$ is equal to the loudness of p_1 and p_2^* played together. In terms of G this condition is

$$G(\alpha^* p_1, 0) = G(p_1, p_2^*). \qquad (13)$$

This equation defines α^*.

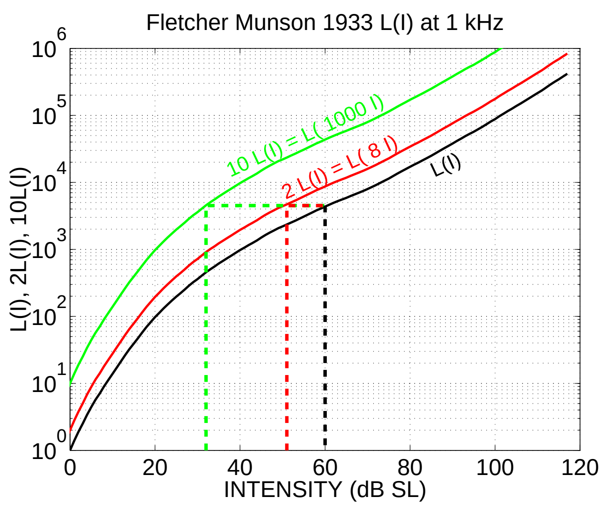

Figure 2. *This figure shows the loudness growth $\mathcal{L}(I)$ (solid line) from Fletcher and Munson (1933) in LU (975 LU is 1 Sone), along with $2\mathcal{L}(I)$ (dashed line) and $10\mathcal{L}(I)$ (dot-dashed line) for reference. To determine $\alpha^*(I)$ draw a horizontal line that crosses the $2\mathcal{L}(I)$ and $\mathcal{L}(I)$ curves, and note the two intensities. The dB-difference is $20\log_{10}(\alpha^*(I))$. For example, one tone at 60 dB SPL is about 4500 LU, and is equal to two equally loud tones played together at 51 dB (i.e., one in each ear). Thus α^* is 9 dB [e.g., $10\log_{10}(8)$]. This is sufficient information to compute the exponent of the loudness power law at 60 dB SL, namely $10\log_{10}(2)/10\log_{10}(8) = 0.335$.*

Results. For f_1 between 0.8 and 8.0 kHz, and f_2 far enough away from f_1 (above or below) so that there is no masking, $20\log_{10}(\alpha^*(I))$ was found to be $\approx$9 dB for p_1 above 40 dB-SL. Below 40 dB-SL, this value decreased linearly to about 2 dB for p_1 at 0 phons, as shown in Fig. 2. It was found that the loudness $G(p_1, p_2^*)$ does not depend on $p_2^*(p_1, f_2)$ as f_2 is varied. Thus we may write $\alpha^*(p_1, \neg p_2^*)$ to show its dependence on p_1 and its *independence* of p_2^*.(Read $\neg p$ as "not dependent on p.")

Fletcher and Munson found an elegant summary of their data. They tested the assumption that

$$G(p_1, p_2) = G(p_1, 0) + G(0, p_2), \qquad (14)$$

namely that the loudnesses of the two tones add. Using Eq. 12, Eq. 14 becomes

$$G(p_1, p_2^*) = 2G(p_1, 0). \tag{15}$$

Combining Eq. 13 and Eq. 15 gives the nonlinear difference equation

$$G(\alpha^*(p_1)p_1, 0) = 2G(p_1, 0), \tag{16}$$

which determines G once $\alpha^*(p_1)$ is specified. $G(p)$ may be found by graphical methods, or by numerical recursion, as shown in Fig. 136, page 190 of Fletcher (1995).

From this formulation Fletcher and Munson found that at 1 kHz, and above 40 dB SPL, the pure-tone loudness G is proportional to the cube root of the signal intensity [$G(p) = (p/p_{\text{ref}})^{2/3}$, since $\alpha^* = 2^{3/2}$, or 9 dB]. This means that if the pressure is increased by 9 dB, the loudness is doubled. Below 40 dB SPL, loudness was frequently approximated as being proportional to intensity [$G(p) = (p/p_{\text{ref}})^2$, $\alpha^* = 2^{1/2}$, or 3 dB]. Figure 2 shows the loudness growth curve. Estimated values of $\alpha^*(I)$ are given in Table 31, page 192, Fletcher (1995).

Loudness JND ΔL. As summarized by Fig. 3, any super-threshold Ψ–domain increments may be quantified using corresponding Φ domain increments. The *loudness JND* $\Delta\mathcal{L}(I)$ is defined as the change in loudness $\mathcal{L}(I)$ corresponding to the intensity JND $\Delta I(I)$. While it is not possible to measure ΔL directly, we assume that we may expand the loudness function in a Taylor series, giving

$$\mathcal{L}(I + \Delta I) = \mathcal{L}(I) + \Delta I \frac{dL}{dI}\bigg|_I + \text{HOT},$$

where HOT represents *higher–order terms*, which we shall ignore. If we solve for

$$\Delta\mathcal{L} \equiv \mathcal{L}(I + \Delta I) - \mathcal{L}(I) \tag{17}$$

we find

$$\Delta L = \Delta I \frac{dL}{dI}\bigg|_I. \tag{18}$$

We call this expression the *small–JND* approximation. The above shows that the loudness JND $\Delta\mathcal{L}(I)$ is related to the intensity JND $\Delta I(I)$ by the slope of the loudness function, evaluated at intensity I. According to the signal detection model, the standard deviation of the single-trial loudness is proportional to the loudness JND, namely $\Delta L = d'\sigma_{\mathcal{L}}$. A more explicit way of expressing this assumption is

$$\frac{\Delta L}{\Delta I} = \frac{\sigma_{\mathcal{L}}}{\sigma_I}. \tag{19}$$

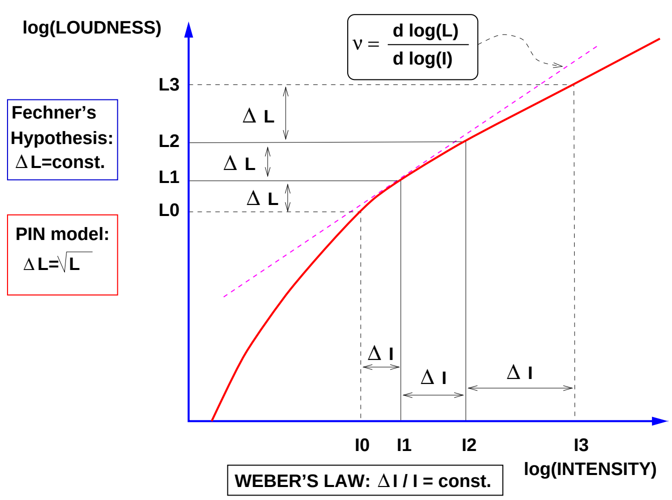

Figure 3. *Relation between ΔI and ΔL as determined by the log-loudness as a function of log-intensity.*

Loudness SNR. In a manner analogous to the Φ–domain SNR_I, we define the Ψ–domain loudness SNR as $\text{SNR}_{\mathcal{L}}(\mathcal{L}) \equiv \mathcal{L}/\sigma_{\mathcal{L}}(\mathcal{L})$. From Fig. 3 it follows that

$$\text{SNR}_I = \nu \text{SNR}_{\mathcal{L}}, \tag{20}$$

where ν is the slope of the log–loudness function with respect to log–intensity, namely

$$\nu(\beta) \equiv \frac{d\mathcal{L}_{\log}}{d\beta}\bigg|_\beta, \tag{21}$$

where $\beta \equiv 10\log_{10}(I/I_{\text{ref}})$ is the *intensity level* in dB, and $\mathcal{L}_{\log}(\beta) \equiv 10\log_{10}(\mathcal{L}(10^{\beta/10}))$. If we express the loudness as a power law $\mathcal{L}(I) = I^{\nu}$, and define $x = \log(I)$, and $y = \log(\mathcal{L})$, then $y = \nu x$. Since the change of ν with respect to dB SPL is small, $dy/dx \approx \Delta y/\Delta x \approx \nu$. Since $d\log(y) = dy/y$,

$$\Delta\mathcal{L}/\mathcal{L} = \nu\Delta I/I. \tag{22}$$

From Eq. 19, Eq. 20 follows. The relationship given in Eq. 22 was first derived c1872 by Plateau, and perhaps it should be called *Plateau's law* [see page lxix of Titchener (1923)].

Equation 20 is important because *(a)* it tells us how to relate the SNRs between the Φ and Ψ domains, *(b)* every term is dimensionless, *(c)* the equation is simple, since ν is approximately constant above 40 dB SL (i.e., Stevens' law), and because *(d)* we are used to seeing and thinking of loudness, intensity, and the SNR, on log scales, and ν as the slope on log-log scales.

Counting JNDs. While the concept of counting JNDs has been frequently discussed in the literature, starting with Fechner, unfortunately the actual counting formula (i.e., the equation) is rarely provided. As a result of a literature search, we found the formula in Nutting (1907), Fletcher (1923a), Wegel and Lane (1924), Riesz (1928), Fletcher (1929), and Miller (1947).

To derive the JND counting formula, Eq. 18 is rewritten as

$$\frac{dI}{\Delta I} = \frac{d\mathcal{L}}{\Delta\mathcal{L}}. \tag{23}$$

Integrating over an interval gives

$$\int_{I_1}^{I_2} \frac{dI}{\Delta I} = \int_{\mathcal{L}_1}^{\mathcal{L}_2} \frac{d\mathcal{L}}{\Delta\mathcal{L}}, \tag{24}$$

where $\mathcal{L}_1 = \mathcal{L}(I_1)$ and $\mathcal{L}_2 = \mathcal{L}(I_2)$. Each integral counts the total number of JNDs between I_1 and I_2.[13,42] For example

$$N_{12} \equiv \int_{I_1}^{I_2} \frac{dI}{\Delta I(I)} \tag{25}$$

defines N_{12}, the number of intensity JNDs between I_1 and I_2. Equivalently

$$N_{12} = \int_{\mathcal{L}_1}^{\mathcal{L}_2} \frac{d\mathcal{L}}{\Delta\mathcal{L}} \tag{26}$$

defines the number of loudness JNDs between $\mathcal{L}_1$ and $\mathcal{L}_2$. The number of JNDs must be the same regardless of the domain (i.e., the abscissa variable), Φ or Ψ.

3. EMPIRICAL MODELS

This section reviews some earlier empirical models of the JND and its relation to loudness relevant to our development.

The Weber fraction. The intensity JND is frequently expressed as a *relative JND* called the *Weber fraction* defined by

$$J(I) \equiv \Delta I(I)/I. \tag{27}$$

From the signal detection theory premise that $\Delta I = d'\sigma_I$,[17] J is just the reciprocal of an effective signal to noise ratio defined as

$$\text{SNR}_I(I) \equiv I/\sigma_I(I) \tag{28}$$

since

$$J = d'\sigma_I/I = d'/\text{SNR}_I. \tag{29}$$

One conceptual problem with the Weber fraction J is that it is an *effective* noise–to–signal ratio, expressed in the Φ (physical) domain, but determined by a Ψ (psychophysical) domain mechanism (internal noise).

Weber's law. In 1846 it was suggested by Weber that $J(I)$ is independent of I. According to Eq. 7,

$$J(I) = 2\alpha_* + \alpha_*^2.$$

If J is constant, then α_* must be constant, which we denote by $\alpha_*(\neg I)$. [As before, $f(\neg I)$ indicates that function f is *not* a function of I]. This expectation, which is called Weber's law,[45] has been successfully applied to many human perceptions. Somewhat frustrating is the empirical observation that $J(I)$ is not constant for the most elementary case of a pure tone.[13,19,3] This observation is referred to as *the near–miss to Weber's law*.[46]

It remains unexplained why Weber's law holds as well as it does,[47,48] or even why it holds at all. Given the nonlinear power law nature of the transformation between the Φ and Ψ domains, coupled with the belief that the noise source is in the Ψ domain, it seems unreasonable that a law as simple as Weber's law, could hold in any general way. A transformation of the JND from the Φ domain to the Ψ domain might clarify the situation. What is needed is the specific dependence of the loudness JND on loudness $\Delta\mathcal{L}(\mathcal{L})$.

Weber's law does make one simple prediction that is potentially important. From Eq. 25 along with Weber's law $J_0 \equiv J(\neg I)$ we see that the formula for the number of JNDs is

$$N_{12} = \int_{I_1}^{I_2} \frac{dI}{J_0 I} = \frac{1}{J_0} \ln(I_2/I_1). \tag{30}$$

Fechner's postulate. In 1860 Fechner postulated that the loudness JND $\Delta\mathcal{L}$ is a constant$^{\|}$.[49–51,7,12] We shall indicate such a constancy with respect to $\mathcal{L}$ as $\Delta\mathcal{L}(\neg\mathcal{L})$. As first reported by Stevens (1961),[52,9] Fechner's postulate is not generally true.

The Fechner JND counting formula. From Eq. 26, along with Fechner's postulate $\Delta\mathcal{L}(\neg\mathcal{L})$, we find

$$N_{12} = \int_{\mathcal{L}_1}^{\mathcal{L}_2} \frac{d\mathcal{L}}{\Delta\mathcal{L}(\neg\mathcal{L})} = \frac{\mathcal{L}_2 - \mathcal{L}_1}{\Delta\mathcal{L}}. \tag{31}$$

This says that if the loudness JND were constant, one could calculate the number of JNDs by dividing the length of the interval by the step size. We call this relation the *Fechner JND counting formula*.

The Weber–Fechner law. It is frequently stated[51] that Fechner's postulate $[\Delta\mathcal{L}(\neg\mathcal{L})]$ and Weber's law $[J_0 \equiv J(\neg I)]$ lead to the conclusion that the difference in loudness between any two intensities I_1 and I_2 is proportional to the logarithm of the ratio of the two intensities, namely

$$\frac{\mathcal{L}(I_2) - \mathcal{L}(I_1)}{\Delta\mathcal{L}} = \frac{1}{J_0} \log(I_2/I_1). \tag{32}$$

This is easily seen by eliminating N_{12} from Eq. 30 and Eq. 31. This result is called *Fechner's law*, and was called the *Weber-Fechner law* by Fletcher and his colleagues because Eq. 32 results when one assumes that both Fechner's postulate and Weber's law are simultaneously true.

Even though Weber's law is approximately true, because Fechner's postulate Eq. 31 is not generally true**,[52] Fechner's law cannot be true. The arguments on both sides of this proposal have been weakened by the unclear relation between loudness and the intensity JND. For example, it has been argued that since the relation between $\mathcal{L}(I)$ and $\Delta I(I)$ depends on many factors, there can be no simple relation between the two.[5] There is a huge literature on the relation between loudness and the JND.[53,54,51,7,12]

Poisson noise. Starting in 1923, Fletcher and Steinberg studied loudness coding of pure tones, noise, and speech,[29,55,28,56] and proposed that loudness was related to neural spike count,[14] and even provided detailed estimates of the relation between the number of spikes and the loudness in sones.[25] In 1943 De Vries first introduced a photon counting Poisson process model as a theoretical basis for the threshold of vision.[57] Siebert (1965) proposed that Poisson–point–process noise, resulting from the neural rate code, acts as the internal noise that limits the frequency JND.[48,19] A few years later,[58] and independently†† McGill and Goldberg (1968a), proposed that the Poisson internal noise (PIN) model might account for the intensity JND, but they did not recover a reasonable loudness growth function. Hellman and Hellman (1990) further refined the argument that Poisson noise may be used to relate the loudness growth to the intensity JND, and they found good agreement between the JND and realistic loudness functions. In 1997 Allen and Neely directly demonstrated that below about 5 sones, $\Delta\mathcal{L}(L) \propto \sqrt{\mathcal{L}}$. This validated McGill and Goldberg's PIN model which assumed that $\sigma_{\mathcal{L}}^2 \propto \mathcal{L}$. Note that the proportionality constant depends on the loudness scale (i.e., sones vs. Fletcher's earlier LU scale) and is therefore not an issue.

The Allen and Neely 1997 estimate of $\Delta\mathcal{L}$ from Riesz's $\Delta I(I)$ and Fletcher and Munson's 1933 $\mathcal{L}(I)$ measurements are described next.

The direct estimate of $\Delta\mathcal{L}$. The loudness $\mathrm{SNR}_{\mathcal{L}}(\mathcal{L}) \equiv \mathcal{L}/\Delta\mathcal{L}(\mathcal{L})$ is computed by dividing $\mathcal{L}(I)$ by $\Delta\mathcal{L}(\mathcal{L})$, as given by Eq. 17. The pure–tone and wideband noise JND results are summarized in terms of the loudness $\mathrm{SNR}_{\mathcal{L}}(I)$ data shown in Fig. 4, where we show $\mathcal{L}/\Delta\mathcal{L}$ = $\mathrm{SNR}_{\mathcal{L}}/d'$ as a function of intensity. The $\mathrm{SNR}_{\mathcal{L}}$ for the wideband noise data of Miller is shown in the lower–left panel as a solid dark line.

Discussion of $\mathrm{SNR}_{\mathcal{L}}$. To the extent that the curves are all approximately the same across frequency, Fig. 4 provides a stimulus independent description of the relation between the intensity JND and loudness. This invariance in $\mathrm{SNR}_{\mathcal{L}}(\mathcal{L})$ seems significant. Where the high level segment of $\mathrm{SNR}_{\mathcal{L}}(\mathcal{L})$ is constant, the intensity resolution of the auditory system has a fixed internal *relative* resolution.[61] The obvious interpretation is that as the intensity is increased from threshold, the neural rate–limited SNR increases until it saturates due to some *other* dynamic range limit, such as that due to some form of central nervous system (CNS) noise.

Near–miss to Stevens' law. In Fig. 5 we show a summary of $\mathcal{L}(I)$, $\nu(I)$, $J(I)$ and $\Delta\mathcal{L}/\mathcal{L} = d'/\mathrm{SNR}_{\mathcal{L}}$ for the tone and noise data. For tones the intensity exponent $\nu(I)$ varies systematically between 0.3 and 0.4 above 50 dB SL, as shown by the solid line in the upper–right panel. We have highlighted this change in the power law with intensity for a 1 kHz tone in the upper–right panel with a light–solid straight line. It is logical to call this effect the *near–miss to Stevens' law*, since over much of the range, it cancels the near–miss to Weber's law, giving a constant relative loudness JND $\Delta\mathcal{L}/\mathcal{L}$ for tones.

The SPIN model. In the lower–right panel we provide a functional summary of $\Delta\mathcal{L}/\mathcal{L}$ for both tones and noise with a light–solid line defined by

$$\frac{\Delta\mathcal{L}(\mathcal{L})}{\mathcal{L}} = h\,[\min(\mathcal{L}, \mathcal{L}_0)]^{-1/2}, \tag{33}$$

where $h = \sqrt{2}$ and $\mathcal{L}_0 = 5000$ LU ($\approx$ 5 sones). We call this relation the *saturated Poisson internal noise* (SPIN) model. With these parameter values, Eq. 33 appears to be a lower bound on the relative loudness $\mathrm{JND}_{\mathcal{L}}$ for both tones and noise.

$^{\|}$We are only considering the auditory case of Fechner's more general theory.

**It may hold in the limited region below 125 Hz and 50 dB SL.

††W. Siebert, personal communication.

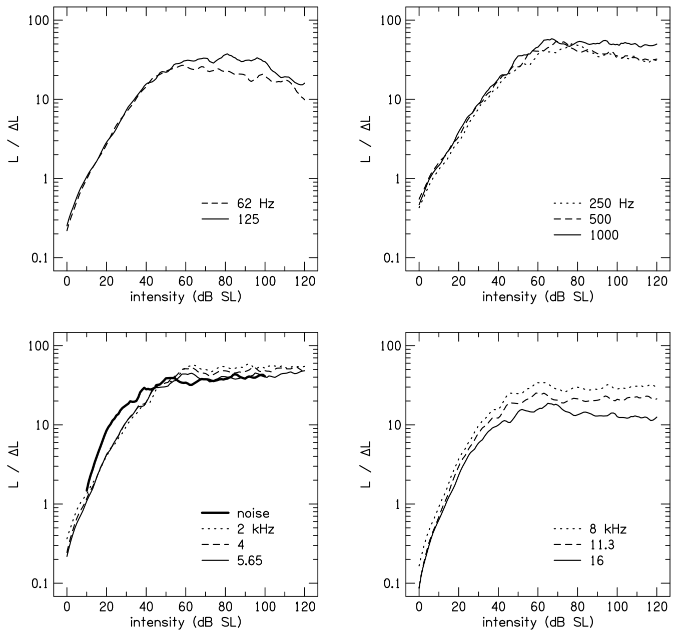

Figure 4. *In this figure we plot $\mathcal{L}(I)/\Delta\mathcal{L}$ = SNR$_\mathcal{L}/d'$ computed directly from Eq. 17 using Riesz's JND data and the Fletcher–Munson loudness–intensity curve, for levels between 0 and 120 dB SL, with frequency as a parameter. Below about 55 dB SL the internal signal to noise ratio SNR$_\mathcal{L}(I)$ is increasing and is approximately proportional to $\mathcal{L}^{1/2}$. Above 60 dB SL the SNR$_\mathcal{L}$ saturates at about 30–50 linear units. For 62 and 125 Hz the SNR$_\mathcal{L}$ slightly decreases at high levels. Results for Miller's noise data is shown as the heavy line in the lower-left panel.*

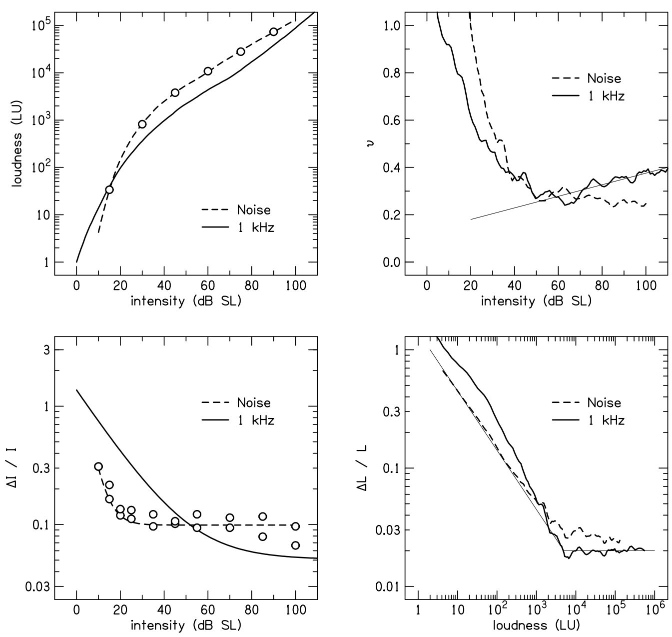

Figure 5. *In 1947 Miller measured the JND$_I$ and the loudness–level for two subjects using wideband noise (0.15–7 kHz) for levels between 3 and 100 dB SL. The intensity of the noise was modulated with a ramped square wave that was high for 1.5 s and low 4.5 s. The loudness, computed from Miller's phon data (dashed curve) using Fletcher and Munson's 1933 1 kHz tone loudness–growth curve are shown in the upper–left panel, along with the Fletcher Munson tonal loudness–growth function (solid curve). The upper–right panel shows the exponent $\nu(I) \equiv d\mathcal{L}_{\log}/d\beta$ for both Fletcher and Munson's and Miller's (average of two subjects) loudness growth function. In the lower–left panel we plot $\Delta I/I$ versus I for Miller's two subjects, Miller's equation, and Riesz's equation. The bottom–right panel shows $\Delta\mathcal{L}/\mathcal{L}$ versus $\mathcal{L}$ for the noise and tones cases. From Eq. 22 $\Delta\mathcal{L}/\mathcal{L} = \nu(I)J(I)$. Note how the product of $\nu(I)$ and $J(I)$ is close to a constant for tones above 65 dB SL. This invariance justifies calling the variations in the power–law exponent $\nu(I)$ for tones the "near–miss to Stevens' law." For reference, 1 sone is 975 LU.*

Weber–fraction formula. Finally we derive the relation between the Weber fraction $J(I)$ given the loudness $\mathcal{L}(I)$ starting from the *small–JND approximation* $\Delta\mathcal{L} = \Delta I \mathcal{L}'(I)$, where $\mathcal{L}'(I) \equiv d\mathcal{L}/dI$. If we solve this equation for ΔI and divide by I we find

$$J(I) \equiv \frac{\Delta I}{I} = \frac{\Delta \mathcal{L}}{I\mathcal{L}'(I)}. \tag{34}$$

Finally we substitute the SPIN model Eq. 33

$$J(I) = \frac{h\mathcal{L}(I)}{I\mathcal{L}'(I)} \min(\mathcal{L}(I), \mathcal{L}_0)^{-1/2} \tag{35}$$

This formula is the same as that derived by Hellman and Hellman (1990) when $\mathcal{L} \leq \mathcal{L}_0$.

Relation Eq. 20 is a simpler, equivalent expression.

4. THE RELATION TO VISION RESEARCH

Cube root compression is required in the cochlea to deal with the limited dynamic range of the cochlear inner hair cells[62] (i.e., < 60 dB) and the limited dynamic range of the central nervous system. There must be similar compression in the eye, for exactly the same reason. Of course the physical source of the compression must be different in the eye.

Superficially speaking, the similarities between auditory and visual psychophysics are impressive. The exponent of 0.3 of the Stevens power law is virtually the same as the auditory case. The spread in the estimated values of each exponent is greater than the magnitude of the differences between the two means – thus the exponents are statistically indistinguishable. There has been an unfortunate and serious confusion regarding the exponents because Stevens unwisely chose to express the loudness exponent in terms of pressure rather than intensity, and did not clearly state what he had done. As a result, many of his summary tables give the loudness exponent as 0.6 and the brightness exponent as 0.3. Yet they are are statistically indistinguishable!

Dynamic range. If we use the intensity JND measurements as a guide to the dynamic range of the ear, we may estimate the ear's dynamic range to be about 10 to 11 orders of magnitude of intensity. The threshold pressure at the eardrum is typically quoted as 14 dB-SPL, while the threshold of pain is close to 120 dB-SPL. This represents a dB difference of 120-14=106 dB, a pressure ratio of $10^{106/20} \approx 10^{5.3}$, or 5.3 orders of magnitude, and an intensity ratio of $10^{106/10} \approx$, or 10.6 orders of magnitude. The estimate of the dynamic range from Riesz's data could be as large as 12 orders of magnitude of intensity. However, the Weber fraction becomes quite large at low intensities; thus 12 orders of magnitude may be an untenable number.

Guided by the intensity JND, the corresponding visual dynamic range is about 8 orders of magnitude of intensity.[2,3]

Transduction Compression. Cube root compression of 9 orders of magnitude dynamic range results in 3 orders of dynamic range magnitude (i.e., a factor of 1000) at the transducer compressor output (of the ear or the eye). Three orders of magnitude is about the dynamic range that can be tolerated by the neural system, at least for the case of the ear. Based on Nyquist–Johnson thermal noise power estimates ($4kTB$), the maximum dynamic range of an inner hair cell (IHC) is about 60 dB.[62] Since the dynamic range of a single auditory nerve fiber is less than 30, the thresholds of many fibers are staggered to code the entire IHC dynamic range. Outer hair cells (OHCs) are responsible for cochlear dynamic range compression and play a key role in determining the Stevens's law exponent.

Dynamic range "recruitment" results from the loss of transducer compression in the auditory system, due to damage of outer hair cells.[40,39,63] Recruitment data taken on subjects with unilateral losses represent a unique opportunity to provide future deep insight into the nature of $\Delta\mathcal{L}(\mathcal{L})$.[64]

The Weber fraction. There has been a long confused history on the meaning of the Weber fraction and its relation to cochlear compression. It is now clear, at least for pure tones and wideband noise, that the intensity JND reflects the internal noise of the neural representation.[9] If the saturation region of the loudness SNR (Ekman fraction) ($\mathcal{L}/\Delta\mathcal{L}$) at 30–50 is due to central noise, as is supposed by Allen and Neely (1997), then we would expect a similar relation for vision. A literature search revealed that just such a proposed relation has been hypothesized by Baird and Noma[11] based on the available data. Baird has further explored this line of reasoning in his more recent book,[12] which he describes as Ekman functions. This analysis is tricky because it is necessary to find loudness/brightness data and intensity JND that are taken under identical conditions. Such data are not always available. Luckily, in the auditory case, they were.

Weber and Fechner were the first to understand the significance of, and attempted to quantify, the Φ-intensity JND. Fechner assumed, incorrectly, that the Ψ-intensity JND was constant. The 1927 work of Thurstone is particularly important to these studies as he was the first to model intensity discrimination and the JND as a random decision variable, leading to the signal detection theory model in psychophysics.

However investigators have failed to focus on the exact relationships. What has been needed is detailed measures and estimates of $\Delta\Psi(\Psi)$, coined the *Ekman function* by Baird (1997). For some reason, these Ekman functions have not been forthcoming. Allen and Neely have found, perhaps for the first time, the SPIN model Eq. 33. The simple physical interpretation of this relation is that the internal noise is Poisson at low intensities, and at high intensities approaches a fixed loudness SNR. The Poisson relationship was predicted in classic papers by Siebert in 1965 and independently by McGill and Goldberg in 1968.

Ψ **additivity.** Fletcher and Munson's 1933 model of loudness was a major advance. However even after being carefully reviewed in 1938 by Stevens and Davis, this important work was ignored by most investigators. Fletcher and Munson introduced many new and important new ideas, including intensity compression, and loudness additivity. Recently it has been shown that additivity also holds for vision.[3] Thus we have failed to build on one of the most potentially important tools in psychophysics, the additivity of the Ψ-intensity. This may be viewed as the "additivity law" or as a basic axiom.[‡‡] By use of the additive property, Fletcher was able to move away from primitive scaling ideas and

[‡‡]If it were to be given a name, I would propose *Fletcher's law*.

build a more quantitative model of the cochlear compression function.[14] This, arguably, allows one to separate the transducer compression from the central properties in a more systematic manner.[41] Given more precise measures of brightness, and intensity JND measurements under identical conditions, it should be possible to define more accurately estimates of the Ekman function, and see if in the visual system it, too, is Poisson.

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