# Why the Disjunction in Quantum Logic is Not Classical*

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**Abstract**

In this paper, the quantum logical 'or' is analyzed from a physical perspective. We show that it is the existence of EPR-like correlation states for the quantum mechanical entity under consideration that make it nonequivalent to the classical situation. Specifically, the presence of potentiality in these correlation states gives rise to the quantum deviation from the classical logical 'or'. We show how this arises not only in the microworld, but also in macroscopic situations where EPR-like correlation states are present. We investigate how application of this analysis to concepts could alleviate some well known problems in cognitive science.

*Dedication:* We dedicate this paper to Marisa Dalla Chiara, one of the founding figures of quantum logic. The ideas expressed in this article have been influenced by her ground-breeaking work in this field.

## 1 Introduction

We put forward a physical explanation of why, in quantum logic, the logical disjunction does not behave classically, even for compatible propositions. Most studies of quantum logic have concentrated on the algebraic structure of the set of propositions, trying to identify the structural differences between quantum logic and classical logic (von Neumann 1932; Birkhoff and von Neumann 1936; Beltrametti and Cassinelli 1981). These mathematical studies, have shown that the quantum logical implication and conjunction can be interpreted as their classical equivalents, while this is not the case for the quantum logical disjunction and negation. We will show that it is the presence of EPR-type quantum mechanical correlations that is at the origin of the nonclassical behavior of the logical disjunction.

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In quantum logic, a proposition $a$ is represented by means of the closed subspace $M_a$ of the Hilbert space $\mathcal{H}$ used to describe the quantum entity under consideration, or by means of the orthogonal projection operator $P_a$ on this closed subspace. We will use both representations, since some of the logical relations in quantum logic can be more easily expressed using the 'closed subspace' representation of a proposition, while others are more easily expressed using the 'orthogonal projection' representation. Let us denote the set of propositions of the quantum entity under consideration by means of $\mathcal{P}$, the set of closed subspaces of the Hilbert space $\mathcal{H}$, describing the quantum entity by means of $\mathcal{L}(\mathcal{H})$ and the set of orthogonal projection operators by means of $\mathcal{P}(\mathcal{H})$. A state $p$ of the quantum mechanical entity under consideration is represented by means of the unit vector $v_p$ of the Hilbert space $\mathcal{H}$.

# 2 The Quantum Logical Operations

For two propositions $a, b \in \mathcal{P}$ the quantum logical operations are introduced by the following expressions:

$$
\begin{aligned}
a \mapsto b \quad &\Leftrightarrow \quad M_a \subset M_b && (1)\\
M_{a \wedge b} \quad &= \quad M_a \cap M_b && (2)\\
M_{a \vee b} \quad &= \quad cl(M_a \cup M_b) && (3)\\
M_{\neg a} \quad &= \quad M_a^{\perp} && (4)
\end{aligned}
$$

We remark that $cl(M_a \cup M_b)$ is the topological closure of the linear space generated by $M_a \cup M_b$. This means that it is the smallest closed subspace of $\mathcal{H}$ that contains $M_a$ and $M_b$.

Using these standard definitions of the quantum logical operations, we can retrieve the 'truth' and 'falseness' of the various possibilities. Suppose that proposition $a \in \mathcal{P}$ is true. This means that the state $p$ of the quantum entity is such that whenever $a$ undergoes a 'yes-no' test $\alpha$, the outcome 'yes' can be predicted with certainty (probability equal to 1). As we know, such a 'yes-no' test in quantum mechanics is represented by the self adjoint operator, the spectral decomposition of which is given by the orthogonal projections $P_a$ and $I - P_a$, where $I$ is the unit operator of the Hilbert space $\mathcal{H}$. From the formalism of quantum mechanics, it follows that proposition $a \in \mathcal{P}$ is true iff the state $p$ of the quantum entity is such that $P_a v_p = v_p$, which is equivalent to $v_p \in M_a$.

## 2.1 The Implication

Let us now consider the quantum logical implication. Suppose we have two propositions $a, b \in \mathcal{P}$ such that $a \mapsto b$, and suppose that $a$ is true. This means that the quantum mechanical entity under consideration is such that for its state $p$ we have $v_p \in M_a$. Since from equation 1 it follows that $M_a \subset M_b$, we have $v_p \in M_b$. This shows that also $b$ is true. This in turn shows that the meaning of $a \mapsto b$ is the following: 'if $a$ is true, then it follows that $b$ is true'.

As a consequence, the quantum logical implication behaves in the same way as the classical logical implication.

### 2.2 The Conjunction

Let us consider the quantum logical conjunction. For two propositions $a, b \in \mathcal{P}$ we consider $a \wedge b$ to be true. This means that the state $p$ of the quantum entity under consideration is such that $v_p \in M(a \wedge b)$. From equation 2, it follows that this is equivalent to $v_p \in M_a \cap M_b$, which is equivalent to $v_p \in M_a$ 'and' $v_p \in M_a$. This again is equivalent to $a$ is true 'and' $b$ is true. Thus we have shown that $a \wedge b$ is true $\Leftrightarrow$ $a$ true 'and' $b$ true. As a consequence, the quantum logical conjunction behaves in the same way as the classical logical conjunction.

### 2.3 The Disjunction

Now we will consider the quantum logical disjunction. For two propositions $a, b \in \mathcal{P}$, we consider $a \vee b$. Let us find out when $a \vee b$ is true. We remark that $M_a \subset cl(M_a \cup M_b)$ and $M_b \subset cl(M_a \cup M_b)$, which shows that $a \mapsto a \vee b$ and $b \mapsto a \vee b$. This means that if $a$ 'or' $b$ is true it follows that $a \vee b$ is true. The inverse implication, however, does not hold. Indeed, $a \vee b$ can be true without $a$ 'or' $b$ being true. The reason is that $cl(M_a \cup M_b)$ contains, in general, vectors that are not contained in $M_a$ or $M_b$. If the quantum entity is in a state $p$ where $v_p$ is such a vector, then $a \vee b$ is true without $a$ or $b$ being true. This shows that the disjunction in quantum logic cannot be interpreted as the disjunction of classical logic.

### 2.4 The Negation

Although it is not the subject of this paper, we can easily see that the quantum logic negation is also not the same as the classical logic negation. Indeed, consider a proposition $a \in \mathcal{P}$ and suppose that $\neg a$ is true. This means that the state $p$ of the considered quantum entity is such that $v_p \in M_a^\perp$. Since $M_a^\perp \cap M_a = \emptyset$ we have that $v_p \notin M_a$, and hence $a$ is not true. This means that if the quantum negation of a proposition is true, then the classical negation of this proposition is true. However the inverse does not hold. In other words, it is possible that the quantum entity is in a state such that $a$ is not true, without $\neg a$ being true.

## 3 EPR-like Correlations and the Nonclassical Nature of Disjunction

As we have shown in the foregoing section, the reason the quantum logical disjunction does not behave classically is that, for two propositions $a, b \in \mathcal{P}$, the quantum entity can be in a state $p$, such that $a \vee b$ is true without $a$ being true or $b$ being true. For such a state $p$ we have that $v_p \in cl(M_a \cup M_b)$, but $v_p \notin M_a$

and $v_p \notin M_b$. We now put forward the main result of this paper: the presence of EPR-like correlations is the origin of the nonclassical nature of the quantum disjunction for the case of compatible propositions.

### 3.1 Compatible Propositions and Truth Tables

Let us now consider two propositions $a, b \in \mathcal{P}$ that are compatible, which means that $P_aP_b = P_bP_a$. In this case, the two 'yes-no' experiments $\alpha$ and $\beta$ that test $a$ and $b$ can be performed together. The experiment that consists of testing the two propositions together, which we denote $\alpha \wedge \beta$, has four possible outcomes (yes, yes), (yes, no), (no, yes) and (no, no). In classical logic, the logical operations can be defined by means of truth tables, and for compatible quantum propositions we can also introduce truth tables. Considering the experiment $\alpha \wedge \beta$ we say that the conjunction $a \wedge b$ is true iff the state of the quantum entity is such that for the experiment $\alpha \wedge \beta$ we obtain with certainty the outcome (yes, yes). Similarly, we say that the disjunction $a \vee b$ is true iff the state of the quantum entity is such that for the experiment $\alpha \wedge \beta$ we obtain with certainty one of the outcomes (yes, yes), (yes, no) or (no, yes).

### 3.2 Compatible Propositions and EPR-like Correlations

Suppose now that we are in a situation where EPR-type correlations exist in relation to the two propositions $a$ and $b$. This means that the state of the quantum entity is such that the measurement $\alpha \wedge \beta$ always leads to the outcome (yes, no) or (no, yes). As a consequence, $a \vee b$ is true. But is is clear that neither $a$ nor $b$ are true in general, which shows that $a$ 'or' $b$ is not true. It is the possibility of such a correlated EPR state that makes the quantum logical disjunction differ from the classical logical disjunction.

## 4 Construction of an EPR-like State for a Quantum Entity

In this section we show that the EPR-like state can be constructed by means of the superposition principle for any two compatible propositions.

Consider $a, b \in \mathcal{P}$ compatible propositions of a quantum entity described in a Hilbert space $\mathcal{H}$, such that $P_a(1-P_b)(\mathcal{H}) \neq \emptyset$ and $(1-P_a)P_b(\mathcal{H}) \neq \emptyset$. A self adjoint operator that corresponds to the measurement of the experiment $\alpha \wedge \beta$ is given by:

$$H = \lambda_1 P_aP_b + \lambda_2 P_a(1-P_b) + \lambda_3(1-P_a)P_b + \lambda_4(1-P_a)(1-P_b) \qquad (5)$$

where $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ are real numbers. The values $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ correspond respectively to the outcomes (yes, yes), (yes, no), (no, yes) and (no, no) of the experiment $\alpha \wedge \beta$. Consider unit vectors $x, y \in \mathcal{H}$ such that $P_a(1-P_b)x = x$ and $(1-P_a)P_by = y$.

We have

$$P_aP_bx = (1-P_a)P_bx = (1-P_a)(1-P_b)x = 0 \tag{6}$$

$$P_aP_by = P_a(1-P_b)y = (1-P_a)(1-P_b)y = 0 \tag{7}$$

Let us indicate how these equalities are derived. For example $P_aP_bx = P_aP_bP_a(1-P_b)x = P_aP_b(1-P_b)x = 0$. The other equalities are derived in an analogous way. Let us consider a state $p$ of the quantum entity such that

$$v_p = \frac{1}{\sqrt{2}}(x+y) \tag{8}$$

We have

$$P_aP_bv_p = (1-P_a)(1-P_b)v_p = 0 \tag{9}$$

$$P_a(1-P_b)v_p = \frac{1}{\sqrt{2}}x \tag{10}$$

$$(1-P_a)P_bv_p = \frac{1}{\sqrt{2}}y \tag{11}$$

Using the quantum formalism and the just derived formulas we can calculate the probabilities, if the quantum entity is in state $v_p$, for a measurement $\alpha \wedge \beta$ to lead to the different outcomes (yes, yes), (yes, no), (no, yes) and (no, no), let us denote them respectively $\mu(\alpha\wedge\beta, yes, yes)$, $\mu(\alpha\wedge\beta, yes, no)$, $\mu(\alpha\wedge\beta, no, yes)$ and $\mu(\alpha\wedge\beta, no, no)$. We have

$$\mu(\alpha\wedge\beta, yes, yes) =< P_aP_bv_p, P_aP_bv_p >= 0 \tag{12}$$

$$\mu(\alpha\wedge\beta, yes, no) =< P_a(1-P_b)v_p, P_a(1-P_b)v_p >= \frac{1}{2} \tag{13}$$

$$\mu(\alpha\wedge\beta, no, yes) =< (1-P_a)P_bv_p, (1-P_a)P_bv_p >= \frac{1}{2} \tag{14}$$

$$\mu(\alpha\wedge\beta, no, no) =< (1-P_a)(1-P_b)v_p, (1-P_a)(1-P_b)v_p >= 0 \tag{15}$$

This proves that for the quantum entity being in state $p$ the experiment $\alpha\wedge\beta$ gives rise to EPR-like correlations for the propositions $a$ and $b$. The possible outcomes are (yes, no) or (no, yes).

## 5 The Quantum 'Or' in the Macroscopic World

Elsewhere in this volume, is a paper that shows that EPR-like correlations also exist for macroscopic entities, depending on the state and propositions that are considered (Aerts et al., 2000). The examples in that paper also shed light on the current subject, so we touch on them again here briefly. For the 'connected vessels of water' example we consider two propositions $a$ and $b$. Proposition $a$ is defined by the sentence: 'there is more than 10 liters of water at the left', and proposition $b$ by the sentence: 'there is more than 10 liters of water at the right'.

The measurement $\alpha$ tests proposition $a$ by pouring out the water at the left with a siphon, and collecting it in a reference vessel, and the measurement $\beta$ does the same at the right. If we test proposition $a$ for the state of the connected vessels containing 20 liters of water, we find that $\alpha$ gives the outcome 'yes' with certainty, and also $\beta$ gives the outcome 'yes' with certainty. If we test the propositions separately, we pour the whole 20 liters out at the left side as well as the right. At first sight this seems to suggest that both propositions are true at once and hence that $a \wedge b$ is true. But after getting a better look we see that this is an error. Indeed, obviously when we pour out the water at the left it influences what happens to the water at the right. More concretely the water at right is also poured out, and hence helps to result in there being more than 10 liters at the left. Indeed, we also know that there cannot be more than 10 liters of water to left and more than 10 liters of water to the right, because the total must equal 20. Our error was to believe that we can test propositions separately in this situation. So let us correct this error by introducing the measurement $\alpha \wedge \beta$ that tests the two propositions together, by pouring out the water at both sides at once. The result is then that if we have more than 10 liters at the left, we have less than 10 liters at the right, and if we have more than 10 liters at the right, we have less than 10 liters at the left. This means that $a \wedge b$ is certainly not true. On the contrary, each time we find $a$ to be true, $b$ turns out not to be true, and vice versa. However, $a \vee b$ is still true, since for $\alpha \wedge \beta$ we always have one of the outcomes (yes, no) 'or' (no, yes). Would this then mean that $a$ 'or' $b$ is true, or equivalently $a$ is true 'or' $b$ is true? Definitely not.

Indeed if $a$ is true 'or' $b$ is true, the measurement $\alpha \wedge \beta$ should give with certainty (yes, yes) or (yes, no), in which case $a$ is true, 'or' it should give with certainty (yes, yes) or (no, yes), in which case $b$ is true. The real situation is more subtle. The connected vessels of water potentially contain 'more than 10 liters of water to the left' 'or' 'more than 10 liters of water to the right', but this potentiality is not made actual before the measurement $\alpha \wedge \beta$ is finished. This is expressed by stating that proposition $a \vee b$ is true. It also shows that $a \vee b$ is not equivalent to $a$ 'or' $b$ as a proposition. It is the possibility of the potential state of the connected vessels of water that makes the 'or' proposition nonclassical.

Let us now turn to the example from cognitive science treated in Aerts et al. 2000. We could restate the insight of the foregoing paragraph for the case of concepts in the mind as follows. We introduce the set of propositions $\{a_n\}$, where $a_n$ is the proposition 'the mind thinks of instance $n$', where each $n$ is an instance of the concept 'cat'. Suppose that the state of the mind is such that it thinks of the concept 'cat'. Just as with the vessels of water, we can say that one of the propositions $a_n$ is true, but only potentially. This is again expressed by the proposition $a_1 \vee a_2 \vee ... \vee a_i \vee ... \vee a_n$ not being equivalent to the proposition $a_1$ 'or' $a_2$ 'or' ... 'or' $a_i$ 'or' ... 'or' $a_n$. Thus we cannot describe a concept as simply a set of instances. It differs from the instances in the same way the connected vessels containing 20 liters of water is different from the set of all separated vessels with water summming to 20 liters. This difference is identical to the well known difference between the electron as described by modern quantum mechanics, and the model that was proposed for the electron in the old quantum theory (before 1926) of a 'cloud' of charged particles inside the atom.

We are currently analyzing how this approach to concepts can shed light on well known problems in cognitive science such as the 'pet fish problem'. Experimental research shows that 'guppy' is not a good example of the concept 'pet', nor is it a good example of the concept 'fish', but it is indeed a good example of the concept 'pet fish' (Osherson and Smith, 1981). Prelimenary investigation indicates that many of the problems that arise with other formal approaches to conceptual dynamics (see Rosch 2000 for a summary) can be resolved using a quantum mechanical approach.

# 6 Conclusion

We have shown how the quantum logical 'or' and its nonequivalence with the classical logical 'or' can be understood from a physical perspective. The origin of the quantum logical 'or' and its difference with the classical logical 'or', is the presence of 'potential correlations' of the EPR-type.

This type of potentiality does not only appear in the microworld, where it is abundant, but also in the macroworld. Special attention has been given to application of this insight to concepts in the mind.

# 7 References


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