K 71379—21/12/2004—RAVICHANDRAN—127843

Received September 2003
Accepted April 2004

QUANTUM PHYSICS

A theory of concepts and their combinations II: A Hilbert space representation

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Abstract
Purpose – To elaborate a theory for modeling concepts that solves the combination problem, i.e. to deliver a description of the combination of concepts. We also investigate the so-called "pet fish problem" in concept research.

Design/methodology/approach – The set of contexts and properties of a concept are embedded in the complex Hilbert space of quantum mechanics. States are unit vectors or density operators and context and properties are orthogonal projections.

Findings – The way calculations are done in Hilbert space makes it possible to model how context influences the state of a concept. Moreover, a solution to the combination problem is proposed. Using the tensor product, a natural product in Hilbert space mathematics, a procedure for describing combined concepts is elaborated. This procedure also provides a solution to the pet-fish problem, and it allows the modeling of a arbitrary number of combined concepts. By way of example, a model for a simple sentence containing a subject, a predicate and an object, is presented.

Originality/value – The combination problem is considered to be one of the crucial unsolved problems in concept research. Also the pet-fish problem has not been solved by earlier attempts of modeling.

Keywords Cybernetics, Mechanical systems, Modelling

Paper type Research paper

1. Introduction
The SCOP theory models a concept as an entity that can be in different states such that a state changes under the influence of a context The notion of "state of a concept" makes it possible to describe a specific contextual effect, namely that an exemplar of the concept has different typicalities and a property of the concept different applicabilities under different contexts. The experiment put forward by Aerts and Gabora (2005) illustrates this contextual effect. In this paper, we present a numerical

The authors would like to thank the 81 friends and colleagues for participating in the experiment presented in this paper, and Alex Riegler and six anonymous reviewers for their comments on the manuscript. This research was supported by Grant G.0039.02 of the Flemish Fund for Scientific Research.

Kybernetes
Vol. 34 No. 1/2, 2005
pp. 176-205

0368-492X
DOI 10.1108/03684920510575807

mathematical model for the representation of a concept, built with a mathematical formalism originally used in quantum mechanics, and we show that the data of the above-mentioned experiment can be reproduced by the model. Specifically, the model is built using the Hilbert space of quantum mechanics, states are represented by unit vectors of this Hilbert space and contexts and properties by projection operators, and the change of state under the influence of a context is described by von Neumann's (1932) "quantum collapse state transformation" in Hilbert space.

This paper deals primarily with the question of what happens when concepts combine. As explained in Aerts and Gabora (2005), known theories of concepts (prototype, exemplar and theory) cannot deliver a model for the description of the combination of concepts. We show that the standard quantum mechanical procedure for the description of the compound of quantum entities, i.e. the tensor product procedure, delivers a description of how concepts combine. Specifically, given the Hilbert spaces of individual concepts, the combination of these concepts is described by the tensor product Hilbert space of these individual Hilbert spaces, and the quantum formalism applied in this tensor product Hilbert space. In this way we work out an explicit description of the combination of "pet" and "fish" in "pet fish", and show that our model describes the guppy effect, and as a consequence solves in a natural way what has come to be known as the "pet fish problem" (Osherson and Smith, 1981, 1982).

We were amazed to find that not only combinations of concepts like "pet fish", but also sentences like "the cat eats the food" can be described in our formalism by nonproduct vectors of the tensor product (representing the so-called entangled states of quantum mechanics) of the individual Hilbert spaces corresponding to the concepts in the combination. It is quantum entanglement that accounts for the most meaningful combinations of concepts. In the last section of the paper we explain the relation between our Hilbert space model of concepts and von Foerster's quantum memory approach.

2. The mathematics for a quantum model

This section introduces the mathematical structure necessary to construct a Hilbert space representation of a concept.

2.1 Hilbert space and linear operators

A Hilbert space $\mathscr{H}$ is a vector space over the set of complex numbers $\mathbb{C}$, in which case we call it a complex Hilbert space, or the set of real numbers $\mathbb{R}$, in which case we call it a real Hilbert space. Thus, the elements of a Hilbert space are vectors. We are interested in finite dimensional complex or real Hilbert spaces and hence do not give a definition of an abstract Hilbert space. Let us denote $\mathbb{C}^n$ to be set of n-tupels $(x_1, x_2, \ldots, x_{n-1}, x_n)$, where each x_k for $1 \leq k \leq n$ is a complex number. In a real Hilbert space, the elements x_k are real numbers, and the set of n-tupels is denoted by $\mathbb{R}^n$. However, we consider the complex Hilbert space case as our default, because the real Hilbert space case is a simplified version of it, and its mathematics follows immediately from the complex case. We define a sum and a multiplication with a complex number as follows. For $(x_1, x_2, \ldots, x_{n-1}, x_n)$, $(y_1, y_2, \ldots, y_{n-1}, y_n) \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, we have:

$$
\begin{aligned}
&(x_1, x_2, \ldots, x_{n-1}, x_n) + (y_1, y_2, \ldots, y_{n-1}, y_n) \\
&\quad = (x_1 + y_1, x_2 + y_2, \ldots, x_{n-1} + y_{n-1}, x_n + y_n)
\end{aligned}
\tag{1}
$$

$$\alpha(x_1, x_2, \ldots, x_{n-1}, x_n) = (\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_{n-1}, \alpha \cdot x_n) \qquad (2)$$

This makes $\mathbb{C}^n$ into a complex vector space. We can call the n tupels $(x_1, x_2, \ldots, x_{n-1}, x_n)$ vectors, and they are denoted as $|x\rangle \in \mathbb{C}^n$. We also define an inproduct between vectors of $\mathbb{C}^n$ as follows. For $|x\rangle, |y\rangle \in \mathbb{C}^n$ we have:

$$\langle x|y\rangle = x^*_1 \cdot y_1 + x^*_2 \cdot y_2 + \cdots + x^*_{n-1} \cdot y_{n-1} + x^*_n \cdot y_n \qquad (3)$$

where x^*_i is the complex conjugate of x_i. Clearly the inproduct of two vectors is a complex number, hence $\langle x|y\rangle \in \mathbb{C}$. For $\alpha, \beta \in \mathbb{C}$ and $|x\rangle, |y\rangle, |z\rangle \in \mathbb{C}^n$ we have:

$$\langle \alpha x + \beta y|z\rangle = \alpha^* \langle x|z\rangle + \beta^* \langle y|z\rangle \qquad (4)$$

$$\langle x|\alpha y + \beta z\rangle = \alpha\langle x|y\rangle + \beta\langle y|z\rangle \qquad (5)$$

This shows that the inproduct is conjugate linear in the first slot, and linear in the second slot of the operation $\langle \cdot | \cdot \rangle$. The complex vector space $\mathbb{C}^n$ equipped with this inproduct is an n-dimensional complex Hilbert space. It is important to mention that any n-dimensional complex Hilbert space is isomorphic to $\mathbb{C}^n$. The inproduct gives rise to a length for vectors and an angle between two vectors, i.e. for $|x\rangle, |y\rangle \in \mathbb{C}^n$ we define:

$$\|x\| = \sqrt{\langle x|x\rangle} \quad \text{and} \quad \cos(x, y) = \frac{|\langle x|y\rangle|}{\|x\| \cdot \|y\|} \qquad (6)$$

Two non-zero vectors $|x\rangle, |y\rangle \in \mathbb{C}^n$ are said to be orthogonal iff $\langle x|y\rangle = 0$. Equation (6) shows that if the inproduct between two non-zero vectors equals zero, the angle between these vectors is 90°. A linear operator A on $\mathbb{C}^n$ is a function $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that

$$A(\alpha|x\rangle + \beta|y\rangle) = \alpha A|x\rangle + \beta A|y\rangle \qquad (7)$$

It can be proven that for the finite dimensional Hilbert space $\mathbb{C}^n$ each linear operator A can be fully described by a $n \times n$ matrix A_{ij}, $1 \leq i \leq n, 1 \leq j \leq n$ of complex numbers, where:

$$A|x\rangle = \left(\sum_{j=1}^{n} A_{1j}x_j, \sum_{j=1}^{n} A_{2j}x_j, \ldots, \sum_{j=1}^{n} A_{n-1,j}x_j, \sum_{j=1}^{n} A_{n,j}x_j\right) \qquad (8)$$

if $|x\rangle = (x_1, x_2, \ldots, x_{n-1}, x_n)$. We make no distinction between the linear operator A and its matrix representation A_{ij}. This gives us the necessary ingredients to explain how states, contexts and properties of a concept are represented in the Hilbert space model.

2.2 States

There are two types of states in quantum mechanics: pure state and density state A pure state is represented by a unit vector $|x\rangle \in \mathbb{C}^n$, i.e. a vector $|x\rangle \in \mathbb{C}^n$ such that $\|x\| = 1$. A density state is represented by a density operator ρ on $\mathbb{C}^n$, which is a linear operator that is self-adjoint. This means that:

$$\rho_{ij} = \rho^*_{ji} \tag{9}$$

for all i, j such that $1 \leq i \leq n, 1 \leq j \leq n$. Furthermore, it is semi definite, which means that $\langle x|\rho|x\rangle \geq 0 \ \forall \, |x\rangle \in \mathbb{C}^n$ and its trace, which is the sum of the diagonal elements of its matrix representation, is equal to 1. Hence

$$\sum_{i=1}^{n} \rho_{ii} = 1.$$

So, to represent the concept "pet" and the situation described previously using this quantum model, we determine the dimension n of the Hilbert space, and represent the states $p_1, p_2, \ldots, p_n \in \Sigma$ of "pet" using unit vectors or density operators of the Hilbert space $\mathbb{C}^n$.

2.3 Properties and weights

A property in quantum mechanics is represented by means of a linear operator, which is an orthogonal projection operator or an orthogonal projector. An orthogonal projection operator P is also a self-adjoint operator; hence equation (9) must be satisfied, i.e. $P_{ij} = P^*_{ji}$. Furthermore for an orthogonal projector, it is necessary that the square of the operator equals the operator itself. Hence $P^2 = P$. Expressed using the components of the matrix of P, this gives

$$\sum_{j=1}^{n} P_{ij} P_{jk} = P_{ik}.$$

This means that to describe the concept "pet" we need to find two orthogonal projection operators P_a and P_b of the complex Hilbert space $\mathbb{C}^n$ that represent the properties $a, b \in \mathcal{L}$.

Let us introduce the quantum mechanical rule for calculating the weights of properties in different states. If the state p is a pure state represented by a unit vector $|x_p\rangle \in \mathbb{C}^n$ we have:

$$\nu(p, a) = \langle x_p|P_a|x_p\rangle \tag{10}$$

If the state p is a density state represented by the density operator ρ_p we have

$$\nu(p, a) = Tr\rho_p P_a \tag{11}$$

where $Tr\rho P_a$ is the trace (the sum of the diagonal elements) of the product of operator ρ with operator P_a.

2.4 Contexts, probabilities and change of state

In quantum mechanics, a measurement is described by a linear operator which is a self-adjoint operator, hence represented by an $n \times n$ matrix M_{ij} that satisfies equation (9), i.e. $M_{ij} = M^*_{ji}$. Although it is standard to represent a context – which in the case of physics is generally a measurement – using a self-adjoint operator, we will use the set of orthogonal projection operators that form the spectral decomposition of this self-adjoint operator, which is equivalent representation. Note that we have been considering "pieces of context" rather than total contexts, and a piece of context is represented by one of these projection operators. Hence, a (piece of) context e is represented by a projector P_e. Such a context e changes a state p of the concept to state q as follows. If p is a pure state represented by the unit vector $|x_p\rangle \in \mathbb{C}^n$ we have:

$$|x_q\rangle = \frac{P_e|x_p\rangle}{\sqrt{\langle x_p|P_e|x_p\rangle}} \tag{12}$$

where

$$\mu(q,e,p) = \langle x_p|P_e|x_p\rangle \tag{13}$$

is the probability that this change takes place. If p is a density state represented by the density operator ρ_p we have:

$$\rho_q = \frac{P_e\rho_p P_e}{Tr\rho_p P_e} \tag{14}$$

where

$$\mu(q,e,p) = Tr\rho_p P_e \tag{15}$$

is the probability that this change takes place.

2.5 Orthonormal bases and superpositions

The representation of a state p by a density operator ρ_p is general enough to include the case of pure states. Indeed, it can be proven that if a density operator is also an orthogonal projector, then it is an orthogonal projector that projects onto one vector.

A set of vectors $B = \{|u\rangle : |u\rangle \in \mathbb{C}^n\}$ is an orthonormal base of $\mathbb{C}^n$ iff

(1) the set of vectors B is a generating set for $\mathbb{C}^n$, which means that each vector of $\mathbb{C}^n$ can be written as a linear combination, i.e. superposition, of vectors of B;

(2) each of the vectors of B has length equal to 1, i.e. $\langle u|u\rangle = 1$ for each $|u\rangle \in B$; and

(3) each two different vectors of B are orthogonal to each other, i.e. $\langle v|w\rangle = 0$ for $|v\rangle, |w\rangle \in B$ and $|v\rangle \neq |w\rangle$.

It can be shown that any orthonormal base of $\mathbb{C}^n$ contains exactly n elements. Given such an orthonormal base B of $\mathbb{C}^n$, any vector $|x\rangle \in \mathbb{C}^n$ can be uniquely written as a linear combination or superposition of the vectors of this base. This means that there exist superposition coefficients $\alpha_u \in \mathbb{C}$ such that

$$|x\rangle = \sum_{|u\rangle\in B}\alpha_u|u\rangle.$$

Using equation (5) we have

$$\langle u|x\rangle = \left\langle u|\sum_{|v\rangle\in B}\alpha_v|v\right\rangle = \sum_{|v\rangle\in B}\alpha_v\langle u|v\rangle = \alpha_u,$$

hence

$$|x\rangle = \sum_{|u\rangle\in B}|u\rangle\langle u|x\rangle \tag{16}$$

From this it follows that

$$\sum_{|u\rangle \in B} |u\rangle\langle u| = 1 \tag{17}$$

which is called the "resolution of this unity" in Hilbert space mathematics. Consider the projector that projects on $|u\rangle$ and denote it P_u. Suppose that $|x\rangle$ is a unit vector. Then we have

$$|x\rangle = \sum_{|u\rangle \in B} P_u |x\rangle.$$

Taking into account equation (16) gives us $P_u = |u\rangle\langle u|$. We also have $P_u|x\rangle = \alpha_u|u\rangle$ and hence

$$\langle x|P_u|x\rangle = \alpha_u \alpha_u^* = |\alpha_u|^2 \tag{18}$$

This proves that the coefficients α_u of the superposition of a unit vector $|x\rangle$ in an orthonormal base B have a specific meaning. From equations (13) and (18) it follows that they are the square root of the probability that the state of the concept represented by $|x\rangle$ changes under the influence of the context represented by P_u.

It is easy to see that the quantum model is a specific realization of a SCOP. Consider the complex Hilbert space $\mathbb{C}^n$, and define

$$\Sigma_Q = \{\rho_p | \rho_p \text{ is a density operator of } \mathscr{H}\},$$

$$\mathscr{M}_Q = \{P_e | P_e \text{ is an orthogonal projection operator of } \mathscr{H}\},$$

$$\mathscr{L}_Q = \{P_a | P_a \text{ is an orthogonal projection operator of } \mathscr{H}\},$$

and the functions μ and ν such that $\mu_Q(q, e, p) = Tr\rho_p Pe$, $\nu_Q(p, a) = Tr\rho_p P_a$ and $\rho_q = P_e \rho_p P_e / Tr\rho_p P_e$, then $(\Sigma_Q, \mathscr{M}_Q, \mathscr{L}_Q, \mu_Q, \nu_Q)$ is a SCOP.

3. A Hilbert space representation of a concept

In this section, we explain how the quantum mechanical formalism is used to construct a model for a concept. We limit ourselves to the construction of a model of "one" concept. In the next section, we explain how it is possible to model combinations of two or more concepts.

3.1 Basic contexts and basic states

Let us re-analyze the experiment in greater detail, taking into account the structure of SCOP derived in Aerts and Gabora (2005). For this purpose, the states and contexts corresponding to the exemplars considered in Table II of Aerts and Gabora (2005) are presented in Table I. So, for example, e_{19} is the context "The pet is a hamster", and p_{15} is the state of "pet" under the context e_{15}, "The pet is a mouse". In the experiment, subjects were asked to estimate the frequency of a specific exemplar of "pet" given a specific context; for example, the exemplar cat for the context e_1, "The pet is chewing a bone", the frequency of the exemplar *dog* for the context e_2, "The pet is being taught", etc. These estimates guide how we embed the SCOP into a Hilbert space. The hypothesis followed in the construction of the embedding is that the frequency

Table I.
States and contexts relevant to exemplars of the concept "pet"

Exemplar	Context	State
Rabbit	E_{13}	p_{13}
Cat	E_{14}	p_{14}
Mouse	E_{15}	p_{15}
Bird	E_{16}	p_{16}
Parrot	E_{17}	p_{17}
Goldfish	E_{18}	p_{18}
Hamster	E_{19}	p_{19}
Canary	E_{20}	p_{20}
Guppy	E_{21}	p_{21}
Snake	E_{22}	p_{22}
Spider	E_{23}	p_{23}
Dog	E_{24}	p_{24}
Hedgehog	E_{25}	p_{25}
Guinea pig	E_{26}	p_{26}

estimates reflect the presence of contexts that are stronger than those explicitly considered in the model, and the distribution of these contexts reflects the frequencies measured in the experiment. Let us call these contexts as *basic contexts*. For example, the contexts:

e_{27}, I remember how I have seen my sister trying to teach her dog to jump over the fence on command (19)

e_{28}, A snake as pet, oh yes, I remember having seen that weird guy on television with snakes crawling all over his body (20)

e_{29}, That is so funny, my friend is teaching his parrot to say my name when I come in (21)

could be such basic contexts. And indeed we have $e_{27} \leq e_2$ and $e_{27} \leq e_{24}$, $e_{28} \leq e_4$ and $e_{28} \leq e_{22}$, and $e_{29} \leq e_5$ and $e_{29} \leq e_{17}$, which shows that these contexts are stronger than any of those considered in the model. Let us denote X the set of such basic contexts for the concept "pet".

Here, we see how our model integrates similarity-based and theory-based approaches. The introduction of this set of contexts might give the impression that basic contexts play somewhat the same role as exemplars play in exemplar models. This is however, not the case; we do not make claims about whether basic contexts are stored in memory. It is possible, for example, that it is a mini-theory that is stored in memory, a mini-theory that has grown out of the experience a subject has had with (part of) the basic contexts, and hence incorporates knowledge about aspects (for example, frequency of appearance in different contexts) of the basic contexts in this way. But it is also possible that some basic contexts are stored in memory. At any rate, they play a structural role in our model, a role related directly to the concept itself. To clarify this, compare their status to the status of a property. The property a_7, *can swim* is a property of the concept "goldfish" independent of the choice of a specific theory of concept representation, or independent of what is or is not stored in memory.

We now introduce some additional hypotheses. First, we suppose that each basic context is an atomic context of $\mathscr{M}$. This means that we stop refining the model with basic contexts; it amounts to demanding that there are no stronger contexts available in the model. They are the most concrete contexts we work with. As mentioned in section 3.5 of Aerts and Gabora (2005), even if a context is an atomic context, there still might be several eigenstates of this context. As an additional hypothesis, we demand that each basic context has only one eigenstate in the model. This means that also on the level of states we want the basic contexts to describe the most refined situation. Indeed, if an atomic context has different eigenstates, the states penetrate more deeply into the refinement of the model than the contexts do. So our demand reflects an equilibrium in fine structure between states and contexts. The set of eigenstates of the atomic contexts we denote U, and we call the elements of U *basic states*. The basic states and contexts are not necessarily possible instances of the concept, but an instance can play the role of a basic state and context. Basic states and contexts can be states and contexts that the subject has been confronted with in texts, movies, dreams, conversations, etc. Let us introduce:

$$E_i = \{u | u \leq e_i, u \in X\} \tag{22}$$

$$X_{ij} = \{u | u \leq e_i \wedge e_j, u \in X\} \tag{23}$$

where E_i is the set of basic contexts that is stronger or equal to e_i, and X_{ij} the set of basic contexts stronger or equal to $e_i \wedge e_j$. It is easy to prove that $X_{ij} = E_i \cap E_j$. Indeed, we have $u \in X_{ij} \Leftrightarrow u \leq e_i \wedge e_j \Leftrightarrow u \leq e_i$ and $u \leq e_j \Leftrightarrow u \in E_i \cap E_j$. Suppose that n is the total number of basic contexts. Let us denote $n(X_{ij})$ the number of basic contexts contained in X_{ij} and $n(E_i)$ the number of basic contexts contained in E_i. We choose $n(X_{ij})$ and $n(E_i)$ as in Table II (we have denoted $n(X_{ij})$ as n_{ij} in Table II).

3.2 Embedding in the Hilbert space

We consider a Hilbert space of dimension 1400, hence $\mathbb{C}^n$, with $n = 1400$. Each basic context $u \in X$ is represented by a projector $|u\rangle\langle u|$, where $|u\rangle \in \mathbb{C}^n$ is a unit vector, and such that $B = \{|u\rangle | u \in X\}$ is an orthonormal base of the Hilbert space $\mathbb{C}^n$, and the corresponding basic state $u \in U$ is represented by this unit vector $|u\rangle \in B$. The ground state $\hat{p}$ of the concept "pet" is represented by a unit vector $|x_{\hat{p}}\rangle$, superposition of the base states $B = \{|u\rangle | u \in X\}$

$$|x_{\hat{p}}\rangle = \sum_{u \in X} \alpha_u |u\rangle \tag{24}$$

where $\alpha_u = \langle u | x_{\hat{p}} \rangle$

$|\alpha_u|^2$ is the probability that the concept "pet" changes to be in base state $|u\rangle$ under context u. We write:

$$|\alpha_u|^2 = \frac{1}{1400} \quad \forall u \in X \tag{25}$$

Table II.
Choice of the distribution of the different types of basic contexts for the concept "pet"

Exemplar	e_1 $n(E_1) = 303$	e_2 $n(E_2) = 495$	e_3 $n(E_3) = 500$	e_4 $n(E_4) = 101$	e_5 $n(E_5) = 200$	e_6 $n(E_6) = 100$	1 $n = 1400$
Rabbit	$n_{13,1} = 12$	$n_{13,2} = 35$	$n_{13,3} = 75$	$n_{13,4} = 5$	$n_{13,5} = 2$	$n_{13,6} = 0$	$n(E_{13}) = 98$
Cat	$n_{14,1} = 75$	$n_{14,2} = 65$	$n_{14,3} = 110$	$n_{14,4} = 3$	$n_{14,5} = 6$	$n_{14,6} = 1$	$n(E_{14}) = 168$
Mouse	$n_{15,1} = 9$	$n_{15,2} = 30$	$n_{15,3} = 40$	$n_{15,4} = 11$	$n_{15,5} = 2$	$n_{15,6} = 0$	$n(E_{15}) = 70$
Bird	$n_{16,1} = 6$	$n_{16,2} = 40$	$n_{16,3} = 10$	$n_{16,4} = 4$	$n_{16,5} = 34$	$n_{16,6} = 1$	$n(E_{16}) = 112$
Parrot	$n_{17,1} = 6$	$n_{17,2} = 80$	$n_{17,3} = 5$	$n_{17,4} = 4$	$n_{17,5} = 126$	$n_{17,6} = 1$	$n(E_{17}) = 98$
Goldfish	$n_{18,1} = 3$	$n_{18,2} = 10$	$n_{18,3} = 0$	$n_{18,4} = 2$	$n_{18,5} = 0$	$n_{18,6} = 48$	$n(E_{18}) = 140$
Hamster	$n_{19,1} = 12$	$n_{19,2} = 35$	$n_{19,3} = 30$	$n_{19,4} = 4$	$n_{19,5} = 2$	$n_{19,6} = 0$	$n(E_{19}) = 98$
Canary	$n_{20,1} = 3$	$n_{20,2} = 35$	$n_{20,3} = 5$	$n_{20,4} = 2$	$n_{20,5} = 14$	$n_{20,6} = 1$	$n(E_{20}) = 112$
Guppy	$n_{21,1} = 3$	$n_{21,2} = 10$	$n_{21,3} = 0$	$n_{21,4} = 2$	$n_{21,5} = 0$	$n_{21,6} = 46$	$n(E_{21}) = 126$
Snake	$n_{22,1} = 6$	$n_{22,2} = 10$	$n_{22,3} = 5$	$n_{22,4} = 22$	$n_{22,5} = 0$	$n_{22,6} = 1$	$n(E_{22}) = 42$
Spider	$n_{23,1} = 3$	$n_{23,2} = 5$	$n_{23,3} = 15$	$n_{23,4} = 23$	$n_{23,5} = 0$	$n_{23,6} = 0$	$n(E_{23}) = 28$
Dog	$n_{24,1} = 150$	$n_{24,2} = 95$	$n_{24,3} = 120$	$n_{24,4} = 3$	$n_{24,5} = 12$	$n_{24,6} = 1$	$n(E_{24}) = 168$
Hedgehog	$n_{25,1} = 6$	$n_{25,2} = 10$	$n_{25,3} = 40$	$n_{25,4} = 12$	$n_{25,5} = 0$	$n_{25,6} = 0$	$n(E_{25}) = 42$
Guinea pig	$n_{26,1} = 9$	$n_{26,2} = 35$	$n_{26,3} = 45$	$n_{26,4} = 4$	$n_{26,5} = 2$	$n_{26,6} = 0$	$n(E_{26}) = 98$

This means that each of the basic states $u \in U$ is considered to have an equal probability of being elicited. We can rewrite the ground state $\hat{p}$ of "pet" more explicitly now:

$$|x_{\hat{p}}\rangle = \sum_{u \in X} \frac{1}{\sqrt{1400}} |u\rangle \tag{26}$$

This means that if the concept "pet" is in its ground state $\hat{p}$, there is a probability of 1/1400 that one of the contexts $u \in X$ acts as a basic context of "pet", and changes the ground state of "pet" to the basic state $u \in U$ of "pet". This means that for "pet" in its ground state, the probability that a basic context that is contained in E_i gets activated and changes the ground state of "pet" to the corresponding basic state, is given by $n(E_i)/1400$, where $n(E_i)$ is given in Table II. Let us show that a straightforward calculation proves that this gives exactly the weights in Table II of Aerts and Gabora (2005). Following Table II in 98 of the 1400 basic contexts, the pet is a hamster. This means that the weight of *hamster* in the ground state of "pet" is $98/1400 = 0.07$, which indeed corresponds with what we find in Table II of Aerts and Gabora (2005) for *hamster*. In 28 of the 1400 basic contexts, the pet is a *spider*. Hence the weight of *spider* in the ground state of "pet" is $28/1400 = 0.02$, as in Table II of Aerts and Gabora (2005). There are 168 of the 1,400 basic contexts where the pet is a *dog*, which means that the weight for *dog* is $168/1400 = 0.12$, as in Table II of Aerts and Gabora (2005).

Now that we have introduced the mathematical apparatus of the quantum model, we can show explicitly how a context changes the state of the concept to another state, and the model remains predicting the data of the experiment. Consider the concept "pet" and the context e_1, "The pet is chewing a bone". The context e_1 is represented by the projection operator P_{e_1} given by:

$$P_{e_1} = \sum_{u \in E_1} |u\rangle\langle u| \tag{27}$$

where E_1 is the set of basic contexts that is stronger than or equal to e_1, hence $E_1 = \{u | u \leq e_1, u \in X\}$. Let us calculate the new state $|x_{p_1}\rangle$ that $|x_{\hat{p}}\rangle$ changes to under the influence of e_1. Following equation (12) we have

$$|x_{p_1}\rangle = \frac{P_{e_1}|x_{\hat{p}}\rangle}{\sqrt{\langle x_{\hat{p}}|P_{e_1}|x_{\hat{p}}\rangle}} \tag{28}$$

Let us calculate this new state explicitly. We have

$$P_{e_1}|x_{\hat{p}}\rangle = \sum_{u \in E_1} |u\rangle\langle u|x_{\hat{p}}\rangle = \sum_{u \in E_1} \frac{1}{\sqrt{1400}}|u\rangle \tag{29}$$

and

$$\langle x_{\hat{p}}|P_{e_1}|x_{\hat{p}}\rangle = \sum_{u \in E_1} \langle x_{\hat{p}}|u\rangle\langle u|x_{\hat{p}}\rangle = \sum_{u \in E_1} |\langle x_{\hat{p}}|u\rangle|^2 = \sum_{u \in E_1} \frac{1}{1400} = \frac{303}{1400} \tag{30}$$

This gives

$$|x_{p_1}\rangle = \sum_{u \in E_1} \frac{1}{\sqrt{303}}|u\rangle \tag{31}$$

3.3 Different states and different weights

We can now show how the quantum model predicts different weights for the contexts corresponding to different exemplars in the experiment. Consider for example, the context e_{14}, "The pet is a cat", and the corresponding state p_{14}, "The pet is a cat", and calculate the probability that p_1 collapses to p_{14} under context e_{14}. First we must calculate the orthogonal projection operator of the Hilbert space that describes e_{14}. This projection operator is given by:

$$P_{e_{14}} = \sum_{u \in E_{14}} |u\rangle\langle u| \tag{32}$$

where $E_{14} = \{u | u \leq e_{14}, u \in X\}$. Following the quantum mechanical calculation in equation (13), we obtain the weight of the exemplar cat under context e_1, i.e. the probability that state p_1 collapses to state p_{14} under context e_{14}, "The pet is a cat". We have

$$\mu(p_{14}, e_{14}, p_1) = \langle x_{p_1}|P_{e_{14}}|x_{p_1}\rangle \tag{33}$$

which gives

$$\langle x_{p_1}|P_{e_{14}}|x_{p_1}\rangle = \sum_{u \in E_{14}} \langle x_{p_1}|u\rangle\langle u|x_{p_1}\rangle = \sum_{u \in E_{14}} \sum_{v \in E_1} \sum_{w \in E_1} \frac{1}{303} \langle v|u\rangle\langle u|w\rangle \tag{34}$$

$$\langle x_{p_1}|P_{e_{14}}|x_{p_1}\rangle = \sum_{u\in E_{14}}\sum_{v\in E_1}\sum_{w\in E_1}\frac{1}{303}\delta(v,u)\delta(u,w) = \sum_{u\in E_1\cap E_{14}}\frac{1}{303} = \frac{75}{303} = 0.25 \tag{35}$$

corresponding with the experimental result in given Table II of Aerts and Gabora (2005). In contrast, let us calculate the weight of the exemplar cat for "pet" in the ground state $\hat{p}$. Applying the same formula (13) we have

$$\mu(p_{14}, e_{14}, \hat{p}) = \langle x_{\hat{p}}|P_{e_{14}}|x_{\hat{p}}\rangle \tag{36}$$

and

$$\langle x_{\hat{p}}|P_{e_{14}}|x_{\hat{p}}\rangle = \sum_{u\in E_{14}}\langle x_{\hat{p}}|u\rangle\langle u|x_{\hat{p}}\rangle = \sum_{u\in E_{14}}\frac{1}{1400} = \frac{168}{1400} = 0.12 \tag{37}$$

This also corresponds to the experimental results given in Table II of Aerts and Gabora (2005).

Let us make some more calculations of states and weights corresponding to exemplars and contexts of the experiment. Consider the context e_6, "The pet is a fish". This context e_6 is represented by the projection operator P_{e_6} given by:

$$P_{e_6} = \sum_{u\in E_6}|u\rangle\langle u| \tag{38}$$

where E_6 is the set of basic contexts that is stronger than or equal to e_6. Hence $E_6 = \{u|u \leq e_6, u \in X\}$. Following equation (12) we obtain the following expression for the state $|x_{p_6}\rangle$

$$|x_{p_6}\rangle = \frac{P_{e_6}|x_{\hat{p}}\rangle}{\sqrt{\langle x_{\hat{p}}|P_{e_6}|x_{\hat{p}}\rangle}} \tag{39}$$

We have

$$P_{e_6}|x_{\hat{p}}\rangle = \sum_{u\in E_6}|u\rangle\langle u|x_{\hat{p}}\rangle = \sum_{u\in E_6}\frac{1}{\sqrt{1400}}|u\rangle \tag{40}$$

and

$$\langle x_{\hat{p}}|P_{e_6}|x_{\hat{p}}\rangle = \sum_{u\in E_6}\langle x_{\hat{p}}|u\rangle\langle u|x_{\hat{p}}\rangle = \sum_{u\in E_6}|\langle x_{\hat{p}}|u\rangle|^2 = \sum_{u\in E_6}\frac{1}{1400} = \frac{100}{1400} \tag{41}$$

This gives

$$|x_{p_6}\rangle = \sum_{u\in E_6}\frac{1}{\sqrt{100}}|u\rangle \tag{42}$$

Suppose we want to calculate the weights of the exemplar "hedgehog" for this state. Again using formula (13) we obtain:

$$\mu(p_{25}, e_{25}, p_6) = \langle x_{p_6} | P_{e_{25}} | x_{p_6} \rangle \tag{43}$$

From Table II follows that $n_{25,6} = 0$, which means that $E_{25} \cap E_6 = \emptyset$. We have no basic contexts in our model where the pet is a fish and a hedgehog. This means that $P_{e_{25}} \perp |x_{p_6}\rangle$, and hence $P_{e_{25}}|x_{p_6}\rangle = |0\rangle$. As a consequence we have $\mu(p_{25}, e_{25}, p_6) = 0$, which corresponds to the experimental result in Table II of Aerts and Gabora (2005).

Let us calculate the weight for the exemplar goldfish in the state p_6. We have:

$$\mu(p_{18}, e_{18}, p_6) = \langle x_{p_6} | P_{e_{18}} | x_{p_6} \rangle \tag{44}$$

where

$$P_{e_{18}} = \sum_{u \in E_{18}} |u\rangle\langle u| \tag{45}$$

and $E_{18} = \{u | u \leq e_{18}, u \in X\}$. Following equation (13) this gives:

$$\langle x_{p_6} | P_{e_{18}} | x_{p_6} \rangle = \sum_{u \in E_{18}} \langle x_{p_6} | u \rangle \langle u | x_{p_6} \rangle = \sum_{u \in E_{18}} \sum_{v \in E_6} \sum_{w \in E_6} \frac{1}{100} \langle v | u \rangle \langle u | w \rangle \tag{46}$$

$$\langle x_{p_6} | P_{e_{18}} | x_{p_6} \rangle = \sum_{u \in E_{18}} \sum_{v \in E_6} \sum_{w \in E_6} \frac{1}{100} \delta(v, u) \delta(u, w) = \sum_{u \in E_{18} \cap E_6} \frac{1}{100} = \frac{48}{100} = 0.48 \tag{47}$$

corresponding with the experimental result given in Table II of Aerts and Gabora (2005).

The foregoing calculations show that our SCOP theory in Hilbert space is able to model the experimental data of the experiment put forward in section 2.2 of Aerts and Gabora (2005). The choice of distribution of the basic contexts and states are presented in Table II, and the corresponding dimension of the Hilbert space is crucial for the model to predict that experimental data. It is possible to see that the distribution of basic contexts and states (Table II) corresponds more or less to a set theoretical model of the experimental data, such that the Hilbert space model can be considered to be a quantization, in the sense used in quantum mechanics, of this set theoretical model.

4. Combinations of concepts in the SCOP model

The previous section explained how to build a model of one concept. This section shows that conceptual combinations can be described naturally using the tensor product of the corresponding Hilbert spaces, the procedure to describe compound entities in quantum mechanics. We give an explicit model for the combinations of the concepts "pet" and "fish", and show that the pet fish problem is thereby solved. Then we illustrate how combinations of more than two concepts can be described. First we need to explain what the tensor product is.

4.1 The tensor product and entanglement

Consider two quantum entities S and T described, respectively, in Hilbert spaces $\mathcal{H}^S$ and $\mathcal{H}^T$. In quantum mechanics there exists a well known procedure to describe the compound $S \otimes T$ of two quantum entities S and T by means of the Hilbert space $\mathcal{H}^S \otimes \mathcal{H}^T$, which is the tensor product of the Hilbert spaces $\mathcal{H}^S$ and $\mathcal{H}^T$. The tensor product behaves like a product; for example, take $\alpha \in \mathbb{C}$, $|x^S\rangle \in \mathscr{H}^S$ and $|x^T\rangle \in \mathscr{H}^T$, then we have

$$\alpha(|x^S\rangle \otimes |x^T\rangle) = (\alpha|x^S\rangle) \otimes |x^T\rangle = |x^S\rangle \otimes (\alpha|x^T\rangle) \tag{48}$$

However, it is not commutative, meaning that even when a Hilbert space is tensored with itself, for $|x\rangle \in \mathscr{H}$ and $|y\rangle \in \mathscr{H}$ we have $|x\rangle \otimes |y\rangle \in \mathscr{H} \otimes \mathscr{H}$ is in general not equal to $|y\rangle \otimes |x\rangle$. The mathematical construction of the tensor product in all its details is not trivial. The best way to imagine what the tensor product space is like is to consider two orthonormal bases B^S and B^T, respectively, of the subspaces $\mathscr{H}^S$ and $\mathscr{H}^T$ and note that the set of vectors $\{|u^S\rangle \otimes |u^T\rangle : |u^S\rangle \in B^S, |u^T\rangle \in B^T\}$ is an orthonormal base of the tensor product $\mathscr{H}^S \otimes \mathscr{H}^T$. Concretely this means that each vector $|z\rangle \in \mathscr{H}^S \otimes \mathscr{H}^T$ can be written as a linear combination of elements of this orthonormal base:

$$|z\rangle = \sum_{|u^S\rangle \in B^S, |u^T\rangle \in B^T} \alpha_{u^S,u^T} |u^S\rangle \otimes |u^T\rangle \tag{49}$$

We need to explain some of the more sophisticated aspects of the tensor product, because they are crucial for the description of conceptual combinations. The first aspect is that vectors of the tensor product can be product vectors or nonproduct vectors. The difference between them can be illustrated with a simple example. Consider the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$, and two vectors $|x\rangle, |y\rangle \in \mathbb{C}^2$, and their tensor product $|x\rangle \otimes |y\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$. Suppose further that $|u\rangle_1$, $|u\rangle_2$ is an orthonormal base of $\mathbb{C}^2$, which means that we can write:

$$|x\rangle = \alpha|u\rangle_1 + \beta|u\rangle_2 \quad \text{and} \quad |y\rangle = \gamma|u\rangle_1 + \delta|u\rangle_2 \tag{50}$$

which gives

$$|x\rangle \otimes |y\rangle = (\alpha|u\rangle_1 + \beta|u\rangle_2) \otimes (\gamma|u\rangle_1 + \delta|u\rangle_2) \tag{51}$$

$$|x\rangle \otimes |y\rangle = \alpha\gamma|u\rangle_1 \otimes |u\rangle_1 + \alpha\delta|u\rangle_1 \otimes |u\rangle_2 + \beta\gamma|u\rangle_2 \otimes |u\rangle_1 + \beta\delta|u\rangle_2|u\rangle_2 \tag{52}$$

Taking into account the uniqueness of the decomposition in equation (49) we have:

$$|x\rangle \otimes |y\rangle = \alpha_{11}|u\rangle_1 \otimes |u\rangle_1 + \alpha_{12}|u\rangle_1 \otimes |u\rangle_2 + \alpha_{21}|u\rangle_2 \otimes |u\rangle_1 + \alpha_{22}|u\rangle_2 \otimes |u\rangle_2 \tag{53}$$

with

$$\alpha_{11} = \alpha\gamma \quad \alpha_{12} = \alpha\delta \quad \alpha_{21} = \beta\gamma \quad \alpha_{22} = \beta\delta \tag{54}$$

It is easy to see that an arbitrary vector $|z\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is not always of the form $|x\rangle \otimes |y\rangle$. For example, choose

$$|z\rangle = |u\rangle_1 \otimes |u\rangle_1 + |u\rangle_2 \otimes |u\rangle_2 \tag{55}$$

This amounts to choosing in the decomposition of $|z\rangle$, following formula (49), $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{12} = \alpha_{21} = 0$. If $|z\rangle$ chosen in this way were equal to a product vector like $|x\rangle \otimes |y\rangle$, we would find $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that equation (54) is satisfied. This means that

$$\alpha\gamma = \beta\delta = 1 \quad \text{and} \quad \alpha\delta = \beta\gamma = 0 \tag{56}$$

This is not possible; there does not exist α, β, γ, δ that satisfy equation (56). Indeed, suppose that $\alpha\delta = 0$, then one of the two α or δ has to equal zero. But then one of the two $\alpha\gamma$ or $\beta\delta$ also equals zero, which proves that they both cannot equal 1, as demanded in equation (56). This proves that $|z\rangle = |u\rangle_1 \otimes |u\rangle_1 + |u\rangle_2 \otimes |u\rangle_2$ is a nonproduct vector, i.e. it cannot be written as the product of a vector in $\mathbb{C}^2$ with another vector in $\mathbb{C}^2$.

Nonproduct vectors of the tensor product Hilbert space represent nonproduct states of the compound concept described by this tensor product Hilbert space. It is these nonproduct states that contain entanglement, meaning that the effect of a context on one of the two sub-entities (sub-concepts) also influences the other sub-entity (sub-concept) in a specific way. As we will see, it is also these nonproduct states that make it possible to represent the relation of entanglement amongst sub-concepts as one of ways concepts can combine. Specifically (as we will show explicitly in Section 4.4) combinations like “pet fish” are described as entangled (nonproduct) states of “pet” and “fish” within the tensor product of their respective Hilbert spaces.

A second aspect of the tensor product structure that must be explained is how projectors work. Projectors enable us to express the influence of context, and how transition probabilities and weights are calculated. Suppose we consider a context $e^S \in \mathscr{M}_S$ of the first concept S, represented by a projection operator P_e^S of the Hilbert space $\mathscr{H}^S$. This context e^S can be considered as a context of the compound $S \otimes T$ of the two concepts S and T, and will then be represented by the projection operator $P_e^S \otimes 1^T$, where 1^T is the unit operator on $\mathscr{H}^T$. If we have a context $e^S \in \mathscr{M}^S$ of the first concept S and a context $e^T \in \mathscr{M}^T$ of the second concept T, represented, respectively, by projection operators P_e^S and P_e^T, then $P_e^S \otimes P_e^T$ represents the context $e^S \otimes e^T$ of the compound concept $S \otimes T$. We have

$$P_e^S \otimes P_e^T(|x^S\rangle \otimes |x^T\rangle) = P_e^S|x^S\rangle \otimes P_e^T|x^T\rangle \tag{57}$$

The transition probabilities and weights are calculated using the following formulas in the tensor product

$$\langle x^S \otimes x^T | y^S \otimes y^T\rangle = \langle x^S|y^S\rangle\langle x^T|y^T\rangle \quad \text{and} \quad Tr(A^S \otimes A^T) = TrA^S \cdot TrA^T \tag{58}$$

A third aspect of the tensor product is the reduced states. If the compound quantum entity $S \otimes T$ is in a nonproduct state $|z\rangle \in \mathscr{H}^S \otimes \mathscr{H}^T$ of the tensor product Hilbert space of the two Hilbert spaces $\mathscr{H}^S$ and $\mathscr{H}^T$ of the sub-entities, then it is not obvious what states the sub-entities are in, because there are no vectors $|x^S\rangle \in \mathscr{H}^S$ and $|x^T\rangle \in \mathscr{H}^T$ such that $|z\rangle = |x^S\rangle \otimes |x^T\rangle$. This means that we can say with certainty that for such a nonproduct state $|z\rangle$, the sub-entities cannot be in pure states. It can be proven in general that the sub-entities are in density states, and these density states are called the reduced states. We do not give the mathematical construction since we only need to calculate the reduced states in specific cases, and refer to Jauch (1968 pp. Chapter 11 Section 7), for a general definition and derivation of the reduced states.

4.2 Combining pet and fish

In this section, we use the quantum formalism to describe how the concepts “pet” and “fish” combine, and see that the “pet fish problem” (Osherson and Smith, 1981, 1982;

Hampton, 1997; Fodor, 1994; Fodor and Lepore, 1996) finds a natural solution (refer Aerts and Gabora (2005) for a presentation of the pet fish problem).

We first have to build the quantum model for the concept "fish", and then combine this, using the tensor product, with the quantum model for "pet". To provide the necessary data, another experiment was performed, using the same subjects and data acquisition methods as for the experiment in Aerts and Gabora (2005). Subjects were asked to rate the frequency of appearance of different exemplars of "fish" under two contexts:

$$e_{30}^{\text{fish}}, \qquad \text{The fish is a pet} \tag{59}$$

and the unity context 1^{fish}. We denote the ground state of "fish" by $\hat{p}^{\text{fish}}$ and the state under context e_{30}^{fish} by p_{30}^{fish}. The results are presented in Table III. We note a similar effect than observed previously for the concept "pet". For example, the weights of *goldfish* and *guppy* are greater under context e_{30}^{fish} than for the ground state under the unity context 1^{fish}, while the weights of all other exemplars are lower.

Let us call X^{fish} the set of basic contexts and U^{fish} the set of basic states that we consider for the concept "fish". We introduce the states and contexts corresponding to the different exemplars that we have considered in the experiment in Table IV. So, for example, the context e_{34}^{fish} is the context "The fish is a dolphin" and the state p_{40}^{fish} is the state of "fish" which is the ground state $\hat{p}^{\text{fish}}$ under the context e_{40}^{fish}, "The fish is a mackerel". Further, we introduce:

$$E_i^{\text{fish}} = \{u | u \leq e_i^{\text{fish}}, u \in X^{\text{fish}}\} \quad \text{and} \quad X_{ij}^{\text{fish}} = \left\{u | u \leq e_i^{\text{fish}} \wedge e_j^{\text{fish}}, u \in X^{\text{fish}}\right\} \tag{60}$$

where E_i^{fish} is the set of basic contexts that is stronger or equal to e_i^{fish} and X_{ij}^{fish} the set of basic contexts that is stronger or equal to $e_i^{\text{fish}} \wedge e_j^{\text{fish}}$. We have $X_{ij}^{\text{fish}} = E_i^{\text{fish}} \cap E_j^{\text{fish}}$. Suppose that m is the total number of basic contexts. Let us denote by $m\left(X_{ij}^{\text{fish}}\right)$ the number of basic contexts contained in X_{ij}^{fish} and by $m\left(E_i^{\text{fish}}\right)$ the number of basic contexts contained in E_i^{fish}. We choose $m\left(X_{ij}^{\text{fish}}\right)$ and $m\left(E_i^{\text{fish}}\right)$ as in Table V. For the

	e_{30}^{fish}		1^{fish}	
Exemplar	Rate	Frequency	Rate	Frequency
Trout	0.54	0.02	4.67	0.09
Shark	0.51	0.02	4.37	0.09
Whale	0.15	0.01	3.36	0.07
Dolphin	0.91	0.04	3.72	0.07
Pike	0.37	0.01	2.94	0.05
Goldfish	6.73	0.40	5.19	0.10
Ray	0.27	0.01	3.10	0.06
Tuna	0.19	0.01	4.57	0.09
Barracuda	0.40	0.01	1.53	0.03
Mackerel	0.19	0.01	3.47	0.07
Herring	0.22	0.39	4.46	0.09
Guppy	6.60	0.01	4.10	0.08
Plaice	0.22	0.05	3.56	0.07
Carp	1.21		3.21	0.06

Table III.
Frequency ratings of different exemplars of the concept "fish" under two contexts

Exemplar	Context	State
Trout	e_{31}^{fish}	p_{31}^{fish}
Shark	e_{32}^{fish}	p_{32}^{fish}
Whale	e_{33}^{fish}	p_{33}^{fish}
Dolphin	e_{34}^{fish}	p_{34}^{fish}
Pike	e_{35}^{fish}	p_{35}^{fish}
Goldfish	e_{36}^{fish}	p_{36}^{fish}
Ray	e_{37}^{fish}	p_{37}^{fish}
Tuna	e_{38}^{fish}	p_{38}^{fish}
Barracuda	e_{39}^{fish}	p_{39}^{fish}
Mackerel	e_{40}^{fish}	p_{40}^{fish}
Herring	e_{41}^{fish}	p_{41}^{fish}
Guppy	e_{42}^{fish}	p_{42}^{fish}
Plaice	e_{43}^{fish}	p_{43}^{fish}
Carp	e_{44}^{fish}	p_{44}^{fish}

Table IV. The states and contexts connected to the exemplars of the concept "fish" that we considered

quantum model of the concept "fish", we consider a Hilbert space $\mathbb{C}^m$ of 408 dimensions.

Let us construct the quantum model for the concept "fish". Each basic context $u \in X^{\text{fish}}$ is represented by a projector $|u\rangle\langle u|$, where $|u\rangle \in \mathbb{C}^m$ is a unit vector, and such that $B^{\text{fish}} = \{|u\rangle | u \in X^{\text{fish}}\}$ is an orthonormal base of the Hilbert space $\mathbb{C}^m$. The basic state corresponding to the basic context u is represented by the vector $|u\rangle$. The ground state $\hat{p}^{\text{fish}}$ of the concept "fish" is represented by the unit vector $\left|x_{\hat{p}}^{\text{fish}}\right\rangle$, superposition of the base states $B^{\text{fish}} = \{|u\rangle | u \in X^{\text{fish}}\}$ using the following expression:

$$\left|x_{\hat{p}}^{\text{fish}}\right\rangle = \sum_{u \in X^{\text{fish}}} \frac{1}{\sqrt{408}} |u\rangle \tag{61}$$

Hence, it the concept "fish" is in its ground state $\hat{p}^{\text{fish}}$ there is a probability of 1/408 that one of the basic states $u \in U^{\text{fish}}$, under contexts $u \in X^{\text{fish}}$, is elicited. This means that for "fish" in its ground state, the probability that a basic state gets elicited corresponding to a context contained in E_i^{fish} is given by $m\left(E_i^{\text{fish}}\right)/408$, where $m\left(E_i^{\text{fish}}\right)$ is given in Table V. A straightforward calculation proves that this gives exactly the weights in Table III. Let us look at some examples. Following Table V, in 20 of the 408 basic contexts, the fish is a pike. This means that the weight of pike in the ground state of "fish" is $20/408 = 0.05$, which indeed corresponds to what we find in Table III for pike. In 28 of the 408 basic contexts, the fish is a dolphin. Hence the weight of dolphin in the ground state of "fish" in $28/408 = 0.07$, as can be found in Table III. In 32 of the 408 basic contexts, the fish is a *guppy*, thus the weight for *guppy* is $32/408 = 0.08$, as in Table III.

Exemplar	e_0^{fish} $m(e_{30}^{\text{fish}}) = 100$	1^{fish} $m = 408$
Trout	$m\left(X_{31,1}^{\text{fish}}\right) = 2$	$m\left(E_{31}^{\text{fish}}\right) = 36$
Shark	$m\left(X_{32,1}^{\text{fish}}\right) = 2$	$m\left(E_{32}^{\text{fish}}\right) = 36$
Whale	$m\left(X_{33,1}^{\text{fish}}\right) = 1$	$m\left(E_{33}^{\text{fish}}\right) = 28$
Dolphin	$m\left(X_{34,1}^{\text{fish}}\right) = 4$	$m\left(E_{34}^{\text{fish}}\right) = 28$
Pike	$m\left(X_{35,1}^{\text{fish}}\right) = 1$	$m\left(E_{35}^{\text{fish}}\right) = 20$
Goldfish	$m\left(X_{36,1}^{\text{fish}}\right) = 40$	$m\left(E_{36}^{\text{fish}}\right) = 40$
Ray	$m\left(X_{37,1}^{\text{fish}}\right) = 1$	$m\left(E_{37}^{\text{fish}}\right) = 24$
Tuna	$m\left(X_{38,1}^{\text{fish}}\right) = 1$	$m\left(E_{38}^{\text{fish}}\right) = 36$
Barracuda	$m\left(X_{39,1}^{\text{fish}}\right) = 1$	$m\left(E_{39}^{\text{fish}}\right) = 12$
Mackerel	$m\left(X_{40,1}^{\text{fish}}\right) = 1$	$m\left(E_{40}^{\text{fish}}\right) = 28$
Herring	$m\left(X_{41,1}^{\text{fish}}\right) = 1$	$m\left(E_{41}^{\text{fish}}\right) = 36$
Guppy	$m\left(X_{42,1}^{\text{fish}}\right) = 39$	$m\left(E_{42}^{\text{fish}}\right) = 32$
Plaice	$m\left(X_{43,1}^{\text{fish}}\right) = 1$	$m\left(E_{43}^{\text{fish}}\right) = 28$
Carp	$m\left(X_{44,1}^{\text{fish}}\right) = 5$	$m\left(E_{44}^{\text{fish}}\right) = 24$

Table V.
Choice of the distribution of the different types of basic contexts for the concept "fish"

Now consider the concept "fish" and the context e_{30}^{fish}, "The fish is a pet". The context e_{30}^{fish} is represented by the projection operator $P_{e_{30}}^{\text{fish}}$ given by:

$$P_{e_{30}}^{\text{fish}} = \sum_{u \in E_{30}^{\text{fish}}} |u\rangle\langle u| \tag{62}$$

where E_{30}^{fish} is the set of basic contexts of "fish" that is stronger than or equal to e_{30}^{fish}, hence $E_{30}^{\text{fish}} = \{u | u \leq e_{30}^{\text{fish}}, u \in X^{\text{fish}}\}$. Let us calculate the new state $\left|x_{p_{30}}^{\text{fish}}\right\rangle$ that $\left|x_{\hat{p}}^{\text{fish}}\right\rangle$ changes to under the influence of e_{30}^{fish}. Following equation (12) we have

$$\left|x_{p_{30}}^{\text{fish}}\right\rangle = \frac{P_{e_{30}}^{\text{fish}}\left|x_{\hat{p}}^{\text{fish}}\right\rangle}{\sqrt{\left\langle x_{\hat{p}}^{\text{fish}}\middle|P_{e_{30}^{\text{fish}}}\middle|x_{\hat{p}}^{\text{fish}}\right\rangle}} \tag{63}$$

We have

$$P_{e_{30}}^{\text{fish}}\left|x_{\hat{p}}^{\text{fish}}\right\rangle = \sum_{u \in E_{30}^{\text{fish}}} |u\rangle\left\langle u \middle| x_{\hat{p}}^{\text{fish}}\right\rangle = \sum_{u \in E_{30}^{\text{fish}}} \frac{1}{\sqrt{408}}|u\rangle \tag{64}$$

and

$$\left\langle x_{\hat{p}}^{\text{fish}} \middle| P_{e_{30}}^{\text{fish}} \middle| x_{\hat{p}}^{\text{fish}} \right\rangle = \sum_{u \in E_{30}^{\text{fish}}} \left\langle x_{\hat{p}}^{\text{fish}} \middle| u \right\rangle \left\langle u \middle| x_{\hat{p}}^{\text{fish}} \right\rangle = \sum_{u \in E_{30}^{\text{fish}}} \left| \left\langle x_{\hat{p}}^{\text{fish}} \middle| u \right\rangle \right|^2 \sum_{u \in E_{30}^{\text{fish}}} \frac{1}{408} = \frac{100}{408} \quad (65)$$

This gives

$$\left| x_{p_{30}}^{\text{fish}} \right\rangle = \sum_{u \in E_{30}^{\text{fish}}} \frac{1}{\sqrt{100}} |u\rangle \quad (66)$$

4.3 The compound pet $\otimes$ fish

The compound of the concepts "pet" and "fish", denoted "pet $\otimes$ fish", is described in the space $\mathbb{C}^n \otimes \mathbb{C}^m$. A specific combination does not correspond to the totality of the new concept "pet $\otimes$ fish", but rather to subset of it. For example, the combination "a pet and a fish" is one subset of states of "pet $\otimes$ fish", and the combination "pet fish" is another. As we will see, "a pet and a fish" corresponds to a subset containing only product states of "pet $\otimes$ fish", while "pet fish" corresponds to a subset containing entangled states of "pet $\otimes$ fish". Let us analyze what is meant by different possible states of the compound "pet $\otimes$ fish" of the concepts "pet" and "fish", hence vectors or density operators of the tensor product Hilbert space $\mathbb{C}^n \otimes \mathbb{C}^m$.

The first state we consider is $\hat{p}^{\text{pet}} \otimes \hat{p}^{\text{fish}}$, the tensor product of the ground state $\hat{p}^{\text{pet}}$ of "pet" and the ground state $\hat{p}^{\text{fish}}$ of "fish", which is represented in $\mathbb{C}^n \otimes \mathbb{C}^m$ by the vector $\left| x_{\hat{p}}^{\text{pet}} \right\rangle \otimes \left| x_{\hat{p}}^{\text{fish}} \right\rangle$. This state is a good representation of the conceptual combination "pet and fish", because for "pet and fish", contexts can act on "pet", or on "fish", or both, and they act independently. More concretely, consider the context e_1^{pet}, "The pet is chewing a bone" acting on the concept "pet". This context, then written like $e_1^{\text{pet}} \otimes 1^{\text{fish}}$, can also act on the "pet" sub-concept of "pet $\otimes$ fish". Then this will just change the ground state $\hat{p}^{\text{pet}}$ of "pet" to state p_1^{pet}, and the ground state $\hat{p}^{\text{fish}}$ of the "fish" sub-concept of "pet $\otimes$ fish" will not be influenced. This is exactly the kind of change that the state represented by $\left| x_{\hat{p}}^{\text{pet}} \right\rangle \otimes \left| x_{\hat{p}}^{\text{fish}} \right\rangle$ entails.

Hence

$$\hat{p}^{\text{pet}} \overset{e_1^{\text{pet}}}{\longmapsto} p_1^{\text{pet}} \Rightarrow \hat{p}^{\text{pet}} \otimes \hat{p}^{\text{fish}} \overset{e_1^{\text{pet}} \otimes 1^{\text{fish}}}{\longmapsto} p_1^{\text{pet}} \otimes \hat{p}^{\text{fish}} \quad (67)$$

$$\hat{p}^{\text{pet}} \overset{e_1^{\text{pet}}}{\longmapsto} p_1^{\text{pet}} \Rightarrow \hat{p}^{\text{pet}} \otimes p_{30}^{\text{fish}} \overset{e_1^{\text{pet}} \otimes 1^{\text{fish}}}{\longmapsto} p_1^{\text{pet}} \otimes p_{30}^{\text{fish}} \quad (68)$$

Similarly, a context that only works on the concept "fish", can work on the "fish" sub-concept of "pet $\otimes$ fish", and in this case will not influence the state of "pet". Hence

$$\hat{p}^{\text{fish}} \overset{e_{30}^{\text{fish}}}{\longmapsto} p_{30}^{\text{fish}} \Rightarrow \hat{p}^{\text{pet}} \otimes \hat{p}^{\text{fish}} \overset{1^{\text{pet}} \otimes e_{30}^{\text{fish}}}{\longmapsto} \hat{p}^{\text{pet}} \otimes p_{30}^{\text{fish}} \quad (69)$$

$$\hat{p}^{\text{fish}} \overset{e_{30}^{\text{fish}}}{\longmapsto} p_{30}^{\text{fish}} \Rightarrow p_i^{\text{pet}} \otimes \hat{p}^{\text{fish}} \overset{1^{\text{pet}} \otimes e_{30}^{\text{fish}}}{\longmapsto} p_i^{\text{pet}} \otimes p_{30}^{\text{fish}} \quad (70)$$

Another state to consider is $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$, represented by the vector $|x_{p_6}^{\text{pet}}\rangle \otimes x_{p_{30}}^{\text{fish}}\rangle$. This is a state where the "pet" is a "fish" and the "fish" is a "pet", hence perhaps this state faithfully represents "pet fish". How can we check this? We begin by verifying different frequencies of exemplars and weights of properties in this state, and seeing whether the guppy effect, described in section 2.1 in Aerts and Gabora (2005), is predicted by the model. Equation (46) gives the calculation for the weight of the exemplar *goldfish* for the concept "pet" in the state p_6^{pet}. Now we calculate the weight for the exemplar *goldfish* for the compound concept "pet ⊗ fish" in the state $p_6^{pet} \otimes p_{30}^{fish}$. Following the quantum mechanical rules outlined in equation (57) we need to apply the projector $P_{e_{18}}^{pet} \otimes 1^{fish}$ on the vector $|x_{p_6}^{pet}\rangle \otimes |y_{p_{30}}^{fish}\rangle$, and use it in the quantum formula (13). This gives:

$$\mu(p_{18}^{pet} \otimes p_{30}^{fish}, e_{18}^{pet} \otimes 1^{fish}, p_6^{pet} \otimes p_{30}^{fish}) = \left(\langle x_{p_6}^{pet}| \otimes \langle x_{p_{30}}^{fish}|\right)\left(P_{e_{18}}^{pet} \otimes 1^{fish}\right)\left(|x_{p_6}^{pet}\rangle \otimes |x_{p_{30}}^{fish}\rangle\right) \tag{71}$$

$$\begin{aligned}\mu(p_{18}^{pet} \otimes p_{30}^{fish}, e_{18}^{pet} \otimes 1^{fish}, p_6^{pet} \otimes p_{30}^{fish}) &= \langle x_{p_6}^{pet}|P_{e_{18}}^{pet}|x_{p_6}^{pet}\rangle\langle x_{p_{30}}^{fish}|x_{p_{30}}^{fish}\rangle \\ &= \langle x_{p_6}^{pet}|P_{e_{18}}^{pet}|x_{p_6}^{pet}\rangle\end{aligned} \tag{72}$$

$$\mu(p_{18}^{pet} \otimes p_{30}^{fish}, e_{18}^{pet} \otimes 1^{fish}, p_6^{pet} \otimes p_{30}^{fish}) = \frac{48}{100} = 0.48 \tag{73}$$

This means that the weight of the exemplar *goldfish* of the sub-concept "pet" of the compound "pet ⊗ fish" in the product state $p_6^{pet} \otimes p_{30}^{fish}$ (the state that represents a "pet ⊗ fish" that is a pet and a fish), is equal to the weight of the exemplar *goldfish* of the concept "pet" in the state p_6^{pet} (the state that represents a pet that is a fish). This is not surprising; it simply means that the tensor product in its simplest type of state, the product state, takes over the weights that were there already for the separate sub-concepts. The guppy effect, identified previously in the states p_6^{pet} of the concept "pet" and p_{30}^{fish} of the concept "fish", remains there in this combination of pet and fish described by this product state $p_6^{pet} \otimes p_{30}^{fish}$. Indeed, we can repeat the calculation of equation (71) on the product state of the ground states – hence the state $\hat{p}^{pet} \otimes \hat{p}^{fish}$ – and find

$$\mu(p_{18}^{pet} \otimes \hat{p}^{fish}, e_{18}^{pet} \otimes 1^{fish}, \hat{p}^{pet} \otimes \hat{p}^{fish}) = \left(\langle x_{\hat{p}}^{pet}| \otimes \langle x_{\hat{p}}^{fish}|\right)\left(P_{e_{18}}^{pet} \otimes 1^{fish}\right)\left(|x_{\hat{p}}^{pet}\rangle \otimes |x_{\hat{p}}^{fish}\rangle\right) \tag{74}$$

$$\begin{aligned}\mu(p_{18}^{pet} \otimes \hat{p}^{fish}, e_{18}^{pet} \otimes 1^{fish}, \hat{p}^{pet} \otimes \hat{p}^{fish}) &= \langle x_{\hat{p}}^{pet}|P_{e_{18}}^{pet}|x_{\hat{p}}^{pet}\rangle\langle x_{\hat{p}}^{fish}|x_{\hat{p}}^{fish}\rangle \\ &= \langle x_{\hat{p}}^{pet}|P_{e_{18}}^{pet}|x_{\hat{p}}^{pet}\rangle\end{aligned} \tag{75}$$

$$\mu(p_{18}^{pet} \otimes \hat{p}^{fish}, e_{18}^{pet} \otimes 1^{fish}, \hat{p}^{pet} \otimes \hat{p}^{fish}) = \frac{140}{1400} = 0.10 \tag{76}$$

We see that the weight of *goldfish* for the sub-concept "pet" of the compound "pet ⊗ fish" equals the weight of *goldfish* for the concept "pet" in the ground state $\hat{p}^{pet}$.

The difference between equations (73) and (76) is the guppy effect in our theory of the compound "pet $\otimes$ fish". It should be stated in the following way. The weight of *goldfish* of the concept "pet" equals 0.10 if "pet" is in its ground state, and equals 0.48 if "pet" is in a state under the context "The pet is a fish". This is the pre-guppy effect identified by introducing contexts for the description of one concept, namely "pet". When "pet" combines with "fish" we get the concept "pet $\otimes$ fish". Now the *guppy* effect manifests in the following way. The weight of *goldfish* for "pet" as a sub-concept of "pet $\otimes$ fish" equals 0.10 if the state of "pet $\otimes$ fish" is such that we have "a pet and a fish" in the state "a pet … and … a fish" (without necessarily the pet being a fish and the fish being a pet, this is the product state of the two ground states, hence $\hat{p}^{\text{pet}} \otimes \hat{p}^{\text{fish}}$). The weight of *goldfish* for "pet" as a sub-concept of "pet $\otimes$ fish" equals 0.48 if the state of "pet $\otimes$ fish" is such that we have "a pet and a fish" in a state where the pet is a fish and the fish is a pet (this is the product state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$). So we get the guppy effect in the combination of the concepts "pet" and "fish". But does this mean that the state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ describes a "pet fish"? The weights of exemplars seem to indicate this, but there is still something fundamentally wrong. Look at formula (71). It reads $\mu(p_{18}^{\text{pet}} \otimes p_{30}^{\text{fish}}, e_{18}^{\text{pet}} \otimes 1^{\text{fish}}, p_6^{\text{pet}} \otimes p_{30}^{\text{fish}})$. This means that under the influence of context $e_{18}^{\text{pet}} \otimes 1^{\text{fish}}$ state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ changes to state $p_{18}^{\text{pet}} \otimes p_{30}^{\text{fish}}$. The state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ is a product state of the compound "pet $\otimes$ fish" where the pet is a fish and the fish is a pet. But if "pet" as sub-concept of the compound collapses to *goldfish* (this is the state transformation $p_6^{\text{pet}} \mapsto p_{18}^{\text{pet}}$), we see that p_{30}^{fish} remains unchanged in the collapse translated to the compound (we have there $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}} \mapsto p_{18}^{\text{pet}} \otimes p_{30}^{\text{fish}}$). This means that the context "The pet is a goldfish" causes "pet" as a sub-concept to collapse to *goldfish*, but leaves "fish" as a sub-concept unchanged. The end state after the collapse is $p_{18}^{\text{pet}} \otimes p_{30}^{\text{fish}}$, which means "a goldfish and a fish" (pet has become goldfish, but fish has remained fish). We could have expected this, because the rules of the tensor product tell us exactly that product states behave this way. Their rules are given in symbolic form in equations (67) and (69). Product states describe combined concepts that remain independent, i.e. the concepts are combined in such a way that the influence of a context on one of the sub-concepts does not influence the other sub-concept. That is why, as mentioned previously, the product states describe the combination with the "and" between the concepts; hence "pet and fish". Then what does the product state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ describe? It describes the situation where the pet is a fish, and the fish is a pet: hence two "pet fish" and not one! And indeed, the mathematics shows us this subtlety. If for two "pet fish", one collapses of *goldfish*, there is not reason at all that the other also collapses to *goldfish*. It might for example, be *goldfish* and *guppy*. So to clarify what we are saying here, a possible instance of state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ of the compound "pet $\otimes$ fish" is "a goldfish and a guppy". Now we can see why this state $p_6^{\text{pet}} \otimes p_{30}^{\text{fish}}$ gives numerical indication of a guppy effect. But we did not really find the guppy effect, for the simple reason that we did not yet identify the state that describes "pet fish" (one unique living being that is a "pet" and a "fish"). It is here that one of the strangest and most sophisticated of all quantum effects comes in, namely entanglement.

4.4 The "Pet Fish" as a quantum entangled state

Consider the context

e_{45}, The pet swims around the little pool where the fish is being fed by the girl (77)

This is a context of "pet" as well as of "fish" It is possible to consider a big reservoir of contexts that have not yet been classified as a context of a specific concept. We denote this reservoir. $\mathscr{M}$. This means concretely that $\mathscr{M}^{\text{pet}} \subset \mathscr{M}$ and $\mathscr{M}^{\text{fish}} \subset \mathscr{M}$. Let us denote $\mathscr{M}^{\text{pet,fish}}$ the set of contexts that are contexts of "pet" and also contexts of "fish". Amongst the concrete contexts that were considered in this paper, there are seven that are elements of $\mathscr{M}^{\text{pet,fish}}$, namely:

$$e_6, e_{18}, e_{21}, e_{30}, e_{36}, e_{42}, e_{45} \in \mathscr{M}^{\text{pet,fish}} \tag{78}$$

We denote $X^{\text{pet,fish}}$ the set of basic contexts that are contexts of "pet" as well as contexts of "fish". We have

$$E_6^{\text{pet}} \subset X^{\text{pet,fish}} \quad \text{and} \quad E_{30}^{\text{fish}} \subset X^{\text{pet,fish}} \tag{79}$$

and to model the concept "pet fish" we make the hypothesis that $E_6^{\text{pet}} = E_{30}^{\text{fish}} = E^{\text{pet,fish}}$, namely that the basic contexts of "pet" where the pet is a fish are the same as the basic contexts of "fish" where the fish is a pet. It is not strictly necessary to hypothesize that these two sets are equal. It is sufficient to make the hypothesis that there is a subset of both that contains the basic contexts of "pet" as well as of "fish" that are also basic context of a pet that is a fish.

We have now everything that is necessary to put forth the entangled state that describes "pet fish". It is the following state:

$$|s\rangle = \sum_{u \in E^{\text{pet,fish}}} \frac{1}{\sqrt{100}} |u\rangle \otimes |u\rangle \tag{80}$$

We claim that this vector represents the state of "pet $\otimes$ fish" that corresponds to the conceptual combination "pet fish". Let us denote it with the symbol s.

Now we have to verify what the states of the sub-concepts "pet" and "fish" are if the compound concept "pet $\otimes$ fish" is in the state s represented by $|s\rangle$. Hence let us calculate the reduced states for both "pet" and "fish" of the state $|s\rangle$. As explained in Section 4.1, for a non-product vector, the reduced states are density operators, not vectors. We first calculate the density operator corresponding to $|s\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$. This is given by:

$$|s\rangle\langle s| = \left(\sum_{u \in E^{\text{pet,fish}}} \frac{1}{\sqrt{100}} |u\rangle \otimes |u\rangle \right) \left(\sum_{v \in E^{\text{pet,fish}}} \frac{1}{\sqrt{100}} \langle v| \otimes \langle v| \right) \tag{81}$$

$$|s\rangle\langle s| = \sum_{u,v \in E^{\text{pet,fish}}} \frac{1}{100} |u\rangle\langle v| \otimes |u\rangle\langle v| \tag{82}$$

We find the two reduced density operators by exchanging one of the two products $|u\rangle\langle v|$ by the inproduct $\langle u|v\rangle$. Taking into account that $\langle u|v\rangle = \delta(u,v)$, we have

$$|s\rangle\langle s|^{\text{pet}} = \sum_{u \in E^{\text{pet,fish}}} \frac{1}{100} |u\rangle\langle u| \quad \text{and} \quad |s\rangle\langle s|^{\text{fish}} = \sum_{u \in E^{\text{pet,fish}}} \frac{1}{100} |u\rangle\langle u| \tag{83}$$

as reduced states for "pet" and "fish", respectively. It is easy to calculate and show that these reduced states behave exactly like the states p_6^{pet} and p_{30}^{fish}, respectively. This means that for influences of contexts and weights of properties limited to one of the two sub-concepts "pet" or "fish", the state $|s\rangle$ behaves exactly as would the product state $|x_{p_6}^{pet}\rangle \otimes |x_{p_{30}}^{fish}\rangle$. This means that as far as the weights of exemplars and properties are concerned, we find the values that have been calculated for the state $|x_{p_6}^{pet}\rangle \otimes |x_{p_{30}}^{fish}\rangle$ in the previous section when the compound concept "pet $\otimes$ fish" is in the entangled state $|s\rangle$.

Let us now see how the state $|s\rangle$ changes under the influence of the context $e_{18}^{pet} \otimes 1^{fish}$, "The pet is a goldfish" of the concept "pet". We have

$$P_{e_{18}}^{pet} \otimes 1^{fish} = \sum_{u \in E_{18}^{pet}} |u\rangle\langle u| \otimes 1 \tag{84}$$

where $E_{18}^{pet} = \{u | u \leq e_{18}^{pet}, u \in X\}$. Hence the changed state of s under the influence of context $e_{18}^{pet} \otimes 1^{fish}$ – let us denote it s' – is given by

$$|s'\rangle = \left(P_{e_{18}}^{pet} \otimes 1^{fish}\right)|s\rangle = \sum_{u \in E_{18}^{pet}} \sum_{v \in E_6^{pet}} |u\rangle\langle u| \otimes 1 \frac{1}{\sqrt{100}} |v\rangle \otimes |v\rangle \tag{85}$$

$$|s'\rangle = \sum_{u \in E_{18}^{pet}} \sum_{v \in E_6^{pet}} \frac{1}{\sqrt{100}} \langle u|v\rangle |u\rangle \otimes |v\rangle = \sum_{u \in E_{18}^{pet}} \sum_{v \in E_6^{pet}} \frac{1}{\sqrt{100}} \delta(u,v) |u\rangle \otimes |v\rangle \tag{86}$$

$$|s'\rangle = \sum_{u \in E_{18}^{pet} \cap E_6^{pet}} \frac{1}{\sqrt{100}} |u\rangle \otimes |u\rangle \tag{87}$$

Calculating the reduced density states gives:

$$|s'\rangle\langle s'|^{pet} = \sum_{u \in E_{18}^{pet} \cap E_6^{pet}} \frac{1}{100} |u\rangle\langle u| \quad \text{and} \quad |s'\rangle\langle s'|^{fish} = \sum_{u \in E_{18}^{pet} \cap E_6^{pet}} \frac{1}{100} |u\rangle\langle u| \tag{88}$$

The reduced state $|s'\rangle\langle s'|^{pet}$ with respect to the concept "pet" is the state of "pet" under the context e_6^{pet}, "The pet is a fish", and the context e_{18}^{pet}, "The pet is a goldfish". This is what we would have expected in any case, because indeed the context e_{18}^{pet}, influences "pet" alone and not "fish". However, the reduced state $|s'\rangle\langle s'|^{fish}$ with respect to the concept "fish" after the change provoked by the context e_{18}^{pet}, "is a goldfish", that only influences the concept "pet" directly, is also a state of "fish" under the context "is a pet" and under the context "is a goldfish". This means that if for "pet fish" the pet becomes a *goldfish*, then also for "fish" the fish becomes a *goldfish*. This is exactly what is described by the entangled state $|s\rangle$ of the tensor product space given in equation (80).

4.5 Combining concepts in sentences

In this section, we apply our formalism to model more than two combinations of concepts. Consider a simple archetypical sentence containing a subject, and object and a predicate connecting both: "The cat eats the food". Three concepts "cat", "eat" and "food" are involved: two nouns and one verb. We want to show that it is possible to represent this sentence as an entangled state of the compound concept "cat ⊗ eat ⊗ food".

We introduce the SCOPs of "cat", "eat" and "food", $(\Sigma^{cat}, \mathcal{M}^{cat}, \mathcal{L}^{cat}, \mu^{cat}, \nu^{cat})$, $(\Sigma^{eat}, \mathcal{M}^{eat}, \mathcal{L}^{eat}, \mu^{eat}, \nu^{eat})$ and $(\Sigma^{food}, \mathcal{M}^{food}, \mathcal{L}^{food}, \mu^{food}, \nu^{food})$. $\mathcal{M}$ is the reservoir of contexts that have not been decided to be relevant for a specific concept, hence $\mathcal{M}^{cat} \subset \mathcal{M}$, $\mathcal{M}^{eat} \subset \mathcal{M}$ and $\mathcal{M}^{food} \subset \mathcal{M}$. We choose Hilbert spaces $\mathcal{H}^{cat}$, $\mathcal{H}^{eat}$ and $\mathcal{H}^{food}$ to represent, respectively, the concepts "cat", "eat" and "food". Then we construct the tensor product Hilbert space $\mathcal{H}^{cat} \otimes \mathcal{H}^{eat} \otimes \mathcal{H}^{food}$ to represent the compound concept "cat ⊗ eat ⊗ food". Consider the three ground states $\left|x_{\hat{p}}^{cat}\right\rangle \in \mathcal{H}^{cat}$, $\left|x_{\hat{p}}^{eat}\right\rangle \in \mathcal{H}^{eat}$ and $\left|x_{\hat{p}}^{food}\right\rangle \in \mathcal{H}^{food}$ of, respectively, "cat", "eat" and "food". The product state $\left|x_{\hat{p}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle \in \mathcal{H}^{cat} \otimes \mathcal{H}^{eat} \otimes \mathcal{H}^{food}$ represents the conceptual combination "cat and eat and food". Although it is technically the simplest combination, the one described by the product state of three ground states of each concept apart, it is rare in everyday life. Indeed, upon exposure to the three concepts "cat" "eat" "food" in a row, the mind seems to be caught in a spontaneous act of entanglement that generates the sentence "the cat eats the food". It is interesting to note that the same phenomenon exists with quantum entities, i.e. separated states get spontaneously entangled under influence of any kind of environment. Let us consider the three concepts "cat", "eats" and "food" connected by the word "and" in a independent, hence non-entangled way; i.e. "cat and eat and food" described by the product state $\left|x_{\hat{p}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle$. Concretely this means that if a specific context influences the concept "cat", then the concepts "eat" and "food" are not influenced. For example, suppose that the ground state $\left|x_{\hat{p}}^{cat}\right\rangle$ of the concept "cat" is changed by the context:

$$e_{46}^{cat}, \qquad \text{The cat is Felix} \tag{89}$$

into the state p_{46}^{cat}, "The cat is Felix". If this context e_{46}^{cat} is applied to the compound concept "cat ⊗ eat ⊗ food" in the product state $\left|x_{\hat{p}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle$, then the compound concept changes state to $\left|x_{p_{46}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle$

$$\left|x_{\hat{p}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle \overset{e_{46}^{cat} \otimes 1^{eat} \otimes 1^{food}}{\longmapsto} \left|x_{p_{46}}^{cat}\right\rangle \otimes \left|x_{\hat{p}}^{eat}\right\rangle \otimes \left|x_{\hat{p}}^{food}\right\rangle \tag{90}$$

This state express "Felix and eat and food" as a state of the compound concept "cat ⊗ eat ⊗ food". Can we determine the state of the compound concept "cat ⊗ eat ⊗ food" that describes the sentence "The cat eats the food"? Again, as in the case of "pet fish" this will be an entangled state of the tensor product Hilbert space. Indeed, for the sentence "The cat eats the food", we require that if, for example, "cat" collapses to "Felix", then also "eat" must collapse to "Felix who eats", and "food" must collapse to "Felix and the food she eats". This means that the sentence "The cat eats the food" is certainly not described by a products state of the tensor product Hilbert space. How do we build the correct entangled state? Let us explain this step-by-step so that we can see how this could work for any arbitrary sentence.

First, we observe that the sentence itself is a context for "cat", "eat" and "food". Let us call it e_{47}, hence

$$e_{47}, \quad \text{The cat eats the food} \tag{91}$$

We have $e_{47} \in \mathscr{M}$, but also $e_{47}^{\text{cat}} \in \mathscr{M}^{\text{cat}}, e_{47}^{\text{eat}} \in \mathscr{M}^{\text{eat}}$ and $e_{47}^{\text{food}} \in \mathscr{M}^{\text{food}}$. Now we introduce $E_{47} = \{u | u \leq e_{47}, u \in X\}$ is the set of basic contexts that are stronger than or equal to e_{47}. The entangled state, element of the tensor product Hilbert space $\mathscr{H}^{\text{cat}} \otimes \mathscr{H}^{\text{eat}} \otimes \mathscr{H}^{\text{food}}$, that describes the sentence "The cat eats the food" is given by:

$$|s\rangle = \sum_{u \in E_{47}} \frac{1}{\sqrt{n(E_{47})}} |u\rangle \otimes |u\rangle \otimes |u\rangle \tag{92}$$

where $n(E_{47})$ is the number of basic contexts contained in E_{47}.

Let us show that this state describes exactly the entanglement of the sentence "The cat eats the food". We calculate the reduced states of "cat", "eat" and "food" when the compound "cat $\otimes$ eat $\otimes$ food" is in the state s represented by $|s\rangle$. We first calculate the density operator corresponding to $|s\rangle$. This is given by:

$$|s\rangle\langle s| = \left(\sum_{u \in E_{47}} \frac{1}{\sqrt{n(E_{47})}} |u\rangle \otimes |u\rangle \otimes |u\rangle \right) \left(\sum_{v \in E_{47}} \frac{1}{\sqrt{n(E_{47})}} \langle v| \otimes \langle v| \otimes \langle v| \right) \tag{93}$$

$$|s\rangle\langle s| = \sum_{u,v \in E_{47}} \frac{1}{n(E_{47})} |u\rangle\langle v| \otimes |u\rangle\langle v| \otimes |u\rangle\langle v| \tag{94}$$

This gives us

$$|s\rangle\langle s|^{\text{cat}} = \sum_{u \in E_{47}^{\text{cat}}} \frac{1}{n\left(E_{47}^{\text{cat}}\right)} |u\rangle\langle u| \tag{95}$$

$$|s\rangle\langle s|^{\text{eat}} = \sum_{u \in E_{47}^{\text{eat}}} \frac{1}{n\left(E_{47}^{\text{eat}}\right)} |u\rangle\langle u| \tag{96}$$

$$|s\rangle\langle s|^{\text{food}} = \sum_{u \in E_{47}^{\text{food}}} \frac{1}{n\left(E_{47}^{\text{food}}\right)} |u\rangle\langle u| \tag{97}$$

as reduced states for "cat", "eat" and "food", respectively. These reduced states behave exactly like the states $p_{47}^{\text{cat}}, p_{47}^{\text{eat}}$ and p_{47}^{food} of, respectively, "cat", "eat" and "food", when it comes to calculating frequency values of exemplars and applicability values of properties.

Let us now see how the state $|s\rangle$ changes under the influence of the context $e_{46}^{\text{cat}} \otimes 1^{\text{eat}} \otimes 1^{\text{food}}$, "The cat is Felix" of the concept "cat" as a sub-concept of the compound concept "cat $\otimes$ eat $\otimes$ food". We have:

$$P^{cat}_{e_{46}} \otimes 1^{eat} \otimes 1^{food} = \sum_{u \in E^{cat}_{46}} |u\rangle\langle u| \otimes 1 \otimes 1 \quad (98)$$

where $E^{cat}_{46} = \{u | u \leq e^{cat}_{46}, u \in X^{cat}\}$. Hence the changed state of s under the influence of context $e^{cat}_{46} \otimes 1^{eat} \otimes 1^{food}$ – let us denote it s' – is given by:

$$\left|s'\right\rangle = \left(P^{cat}_{e_{46}} \otimes 1^{eat} \otimes 1^{food}\right)\left|s\right\rangle = \sum_{u \in E^{cat}_{46}} \sum_{v \in E_{47}} |u\rangle\langle u| \otimes 1 \otimes 1 \frac{1}{\sqrt{n(E_{47})}} |v\rangle \otimes |v\rangle \otimes |v\rangle \quad (99)$$

$$|s'\rangle = \sum_{u \in E^{cat}_{46}} \sum_{v \in E_{47}} \frac{1}{\sqrt{n(E_{47})}} \langle u|v\rangle |u\rangle \otimes |v\rangle \otimes |v\rangle$$

$$= \sum_{u \in E^{cat}_{46}} \sum_{v \in E_{47}} \frac{1}{\sqrt{n(E_{47})}} \delta(u, v) |u\rangle \otimes |v\rangle \otimes |v\rangle \quad (100)$$

$$|s'\rangle = \sum_{u \in E^{cat}_{46} \cap E_{47}} \frac{1}{\sqrt{n(E_{47})}} |u\rangle \otimes |u\rangle \otimes |u\rangle \quad (101)$$

Calculating the reduced density states gives:

$$|s'\rangle\langle s'|^{cat} = \sum_{u \in E^{cat}_{46} \cap E^{cat}_{47}} \frac{1}{n(E_{47})} |u\rangle\langle u| \quad (102)$$

$$|s'\rangle\langle s'|^{eat} = \sum_{u \in E^{cat}_{46} \cap E^{eat}_{47}} \frac{1}{n(E_{47})} |u\rangle\langle u| \quad (103)$$

$$|s'\rangle\langle s'|^{food} = \sum_{u \in E^{cat}_{46} \cap E^{food}_{47}} \frac{1}{n(E_{47})} |u\rangle\langle u| \quad (104)$$

The reduced state $|s'\rangle\langle s'|^{cat}$ with respect to the concept "cat" is the state of "cat" under the context $e^{cat}_{46} \wedge e_{47}$, "The cat is Felix and the cat eats the food". This is what we would have expected in any case, because indeed the context $e^{cat}_{46} \otimes 1^{eat} \otimes 1^{food}$ influences "cat" alone and not "eat" and "food". However, the reduced state $|s'\rangle\langle s'|^{eat}$ with respect to the concept "eat" after the change provoked by the context $e^{cat}_{46} \otimes 1^{eat} \otimes 1^{food}$, "The cat is Felix", that only influences "cat" directly, is also a state of "eat" under the context $e^{cat}_{46} \wedge e_{47}$, "The cat is Felix and the cat eats the food", hence "Felix eats the food". This means that if for "The cat eats the food" the "cat" becomes "Felix", then also "eat" becomes "Felix who eats". A similar phenomenon happens for the concept "food". The reduced state $|s'\rangle\langle s'|^{food}$ after the change provoked by the context $e^{cat}_{46} \otimes 1^{eat} \otimes 1^{food}$, "The cat is Felix", that only influences "cat" directly, is also a state of "food" under the context $e^{cat}_{46} \wedge e_{47}$, "The cat is Felix and the cat eats the food", hence “Felix eats the food”. This means that if for “The cat eats the food” the “cat” becomes “Felix”, then also “food” becomes “Felix who eats the food”.

The approach that we have put forward in this paper can be used to elaborate the vector space models for representing words that are used in semantic analysis. The tensor product, and the way that we introduced entangled states to represent sentences, can be used to “solve” the well known “bag of word” problem (texts are treated as “bag of words”, hence order and syntax cannot be considered) as formulated in semantic analysis (Aerts and Czachor, 2004). In a forthcoming paper we investigate more directly how the quantum structures introduced in Aerts and Gabora (2005), i.e. the complete orthocomplemented lattice structure, can be employed in semantic analysis models, and also the relation of our approach with ideas formulated in Widdows (2003) and Widdows and Peters (2003) about quantum logic and semantic analysis.

4.6 A quantum theory of memory

Von Foerster (1950) develops a theory of memory and hints to show how a quantum mechanical formalism could be used to formalize his theory. Von Foerster was inspired by how quantum mechanics was introduced in biology. Genes, the carriers of heredity, are described as quantized states of complex molecules. Von Foerster introduces what he calls *carriers of elementary impressions*, which he calls *mems*, to stress the analogy with *genes*, and introduces the notion of *impregnation* as an archetypical activation of a carrier by an impression. Such an *impregnation of a mem* is formalized as a *quantum mechanical excitation* of one energy level of the mem to another energy level of this same mem, in analogy how this happens with a molecule. A molecule in an excited state spontaneously falls back to a lower energy state, and this process is called decay. The decay process of a mem in a high level energy state to a lower level energy state describes the phenomenon of *forgetting*. The introduction of the quantum mechanical mechanism of excitation and decay between different energy levels of a mem as the fundamental process of memory, respectively, accounting for the learning and the forgetting process, is not developed further in von Foerster’s publication. Von Foerster’s conviction about the relevance of quantum mechanics to memory comes from his phenomenological study of the dynamics of the forgetting process. Although not very explicit about this aspect, it can be inferred from his paper that in his opinion the physical carrier of the mem is a molecule in the brain, such as a large protein, and that memory is hence stored within a micro-physical entity, entailing quantum structure because of its micro-physical nature.

The theory of concepts that we have elaborated is in some respects quite different from von Foerster’s approach, but in other respects can deliver a possible theoretical background for this approach. It is different since we do not believe it to be necessary that there need to be a micro-physical carrier for the quantum structure identified in SCOP. It is not excluded that the quantum structure is encrypted in a quite unique way in the brain, making use of the possibility to realize quantum structure in the macro-world, without the need of micro-physical entities (Aerts, 1982, 1985; Aerts and Van Bogaert, 1992; Aerts *et al.*, 1993, n.d., 1994). On the other hand, if micro-physical entities in the brain serve as carriers of quantum mechanical structure, our SCOP theory could provide specific information about this structure. We can also now clarify the notion of ground state. If a concept is not evoked in any specific kind of way, which is equivalent to it being under the influence of the bath of all types of contexts that can evoke it, we consider it to be in its ground state. Here, we align our theory with von Foerster's idea and use the quantum mechanical processes of excitation and decay to point out specific influences of contexts on the state of a concept. If the concept "pet", changes to the state p_1 under the influence of context e_1, "The pet is chewing a bone", then p_1 is an excited state with respect to the ground state $\hat{p}$ of "pet". The state p_1 will spontaneously decay to the ground state $\hat{p}$. We "forget" after a little while the influence of context e_1, "The pet is chewing a bone" on the concept "pet" and consider "pet" again in its ground state when a new context arrives that excites it again to another state. The process of excitation and de-excitation or decay, goes on in this way, and constitutes the basic dynamics of a concept in interaction with contexts. This is very much aligned with what von Foerster intuitively had in mind in von Foerster (1950), and fits completely with a further quantum mechanical elaboration of our SCOP theory of concepts. It is worth mentioning further steps that can be taken in this direction, although they are speculative, since it shows some of the possible perspectives that can be investigated in future research. If a molecule de-excites (or decays) and collapses to its ground state (or to a lower energy state) it sends out a photon exactly of the amount of energy that equals the difference between the energy of the ground state (the lower energy state) and the excited state. This restores the energy balance, and also makes the quantum process of de-excitation compatible with the second law of thermodynamics. Indeed, a lower energy state is a state with less entropy as compared to a higher energy state, and the ground state is the least entropy state. This means that the decrease of entropy by de-excitation has to be compensated, and this happens by the sending out of the photon that spreads out in space, and in this way increases the entropy of the compound entity molecule and photon. The entropy reasoning remains valid for the situation that we consider, independent of whether we suppose that the quantum structure in the mind is carried by micro-physical entities or not. This means that a de-excitation, e.g. the concept "pet" that in state p_1 decays to the ground state $\hat{p}$, should involve a process of spreading out of a conceptual entity related to "pet". Our speculation is that speech, apart from the more obvious role it plays in communication between different minds, also fulfills this role. This is probably the reason that if the de-excitation is huge and carries a big emotional energy, speech can function as a catharsis of this emotional energy, which would be why psychotherapy consisting of talking can function quite independent of the content of what is said.

The global and speculative view that can be put forward is the following. The compound of all concepts relevant to a certain individual are stored in memory (a more correct way to say this would be: they are memory) and one specific state of mind of the individual will determine one specific state of this compound of concepts. This state of the compound of concepts is a hugely entangled state, but such that most of the time, the reduced states for each concept apart are the ground states. Any specific context will influence and change the state of mind of the individual, and hence also the entangled state of the compound of concepts, and hence also the ground states of some of the individual concepts. These are the concepts that we will identify as being evoked by this specific context. Most of these changes of state are just excitations that spontaneously will de-excite, such that all the individual concepts are in their ground states again. From time to time however, a change of state will have consequences that change the structure of the entanglement, or even the structure of some of the concepts themselves. This are the times that the individual learns something new that will be remembered in his or her long-term memory, and that will provoke a change of his or her world views. The energetic balance gets redefined when this happens, and a new stable entangled state of the compound of all concepts is introduced, giving rise to new ground states for the individual concepts (for example, pets are no longer seen as they used to be once one has his or her own pet). This new situation, just as the earlier one, is again open to influences of contexts that introduce again the dynamics of excitation and spontaneous de-excitation.

5. Summary and conclusions

Von Foerster was inclined to push the formalization of whatever happened to interest him at a given time as far as it could go using whatever tools did the job in order to penetrate into the phenomenon more deeply. In this paper, we take a non-operational step, embedding the SCOP in a more constrained structure, the complex Hilbert space, the mathematical space used as a basis of the quantum mechanical formalism. We have good reasons to do so. The generalized quantum formalisms entail the structure of a complete orthocomplemented lattice, and its concrete form, standard quantum mechanics, is formulated within a complex Hilbert space. The SCOP representation of a concept thereby makes strong gains in terms of calculation and prediction power, because it is formulated in terms of the much less abstract numerical space, the complex Hilbert space.

Section 2 outlines the mathematics of a standard quantum mechanical model in a complex Hilbert space. It is not only the vector space structure of the Hilbert space that is important but also the quantum way of using the Hilbert space. A state is described by a unit vector or a density operator, and a context or property by an orthogonal projection. The quantum formalism furthermore determines the formulas that describe the transition probabilities between states and the weights of the properties. It is by means of these probabilities and weights that we model the typicality values of exemplars and applicability values of properties.

In Section 3, we embed the SCOP in a complex Hilbert space, and call the resulting model “the quantum model of a concept”, to distinguish it from the more abstract SCOP model. The quantum model is similar to a SCOP model, but it is more precise and powerful because it allows specific numerical predictions. We represented the exemplars, contexts, and states that were tested experimentally for the concept “pet”. Each exemplar is represented as a state of the concept. The contexts, states and properties considered in the experiment are embedded in the complex Hilbert space, where contexts figure as orthogonal projections, states as unit vectors or density operators, and properties as orthogonal projections. The embedding is faithful in the sense that the predictions about frequency values of exemplars and applicability values of properties of the model coincide with the values yielded by the experiment (Section 3.3).

Notice how the so-called “pet fish problem” disappears in our formalism. The pet fish problem refers to the empirical result that a guppy is rated as a good example, not of the concept “pet”, nor of the concept “fish”, but of the conjunction “pet fish”. This phenomenon that the typicality of the conjunction is not a simple function of the typicality of its constituent, has come to be known as the “guppy effect”, and it cannot be predicted or explained by contemporary theories of concepts. In our experiment, and hence also in the quantum model, we have taken the context "The pet is a fish" to be a context of the concept "pet". Both experiment and quantum model description show the guppy effect appearing in the state of "pet" under the context "The pet is a fish". Subjects rate guppy as a good example of "pet" under the context. "The pet is a fish", and not as a good example of "pet", and the ratings are faithfully described by the quantum model (Section 3.3). Of course this is not the real guppy effect, because we did not yet describe the combination of the concept "pet" and "fish". Section 4 is devoted to modeling concept combination.

A specific procedure exists to describe the compound of two quantum entities. The mathematical structure that is used is the structure of the tensor product of the Hilbert spaces that are used to describe the two sub-entities. Section 4.1 outlines the tensor product procedure for quantum entities. The tensor product of Hilbert spaces is a sophisticated structure. One of its curious properties is that it contains elements that are called non-product vectors. The states described in quantum mechanics by these non-product vectors of the tensor product of two Hilbert spaces are the so-called "entangled quantum states". They describe entanglement between two quantum entities when merging with each other to form a single compound. In the process of working on this quantum representation of concepts, we were amazed to find that it is these very non-products states that describe the most common combinations of concepts, and that more specifically a "pet fish" is described by entangled states of the concepts "pet" and "fish". This enables us to present a full description of the conceptual combination "pet fish" and hence a solution to the pet fish problem in Section 4.4. There is more to the tensor product procedure than combining concepts. For example, it allows the modeling of combinations of concepts such as "a pet and a fish", something completely different from "pet fish". In this case, product states are involved, which means that the combining of concepts by using the word "and" does not entail entanglement (Section 4.3). Finally, we show how our theory makes it possible to describe the combination of an arbitrary number of concepts, and work out the concrete example of the sentence "The cat eats the food" (Section 4.5).

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