A Boundary-Sensitive Approach to Qualitative Location

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Reasoning about the location of regions in 2-dimensional space is necessarily based on finite approximations to such regions. These finite approximations are often derived by describing how a region (the figure) relates to a frame of reference (the ground). The frame of reference generally consists of regions, or cells, forming a partition of the space under consideration. This paper presents a new approach to describing figure-ground relationships which is able to take account of how the figure relates to boundaries between cells as well as to their interiors. We also provide a general theory of how approximations to regions lead to approximations to operations on regions. This theory is applied to the case of our boundary-sensitive model of location. The paper concludes by indicating how interpreting boundaries in a more general sense should lead to a theory dealing with generalized partitions in which the cells may overlap. The applications of the theory developed here will include qualitative spatial reasoning, and should have practical relevance to geographical information systems.

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1. Introduction

Spatial data is usually presented in terms of numerical coordinates. This allows the expression of precise facts about the world, but has some significant limitations. One limitation is that the representation provided is far removed from the high-level conceptual information involved in many tasks which are generally carried out effectively by humans without access to numerical coordinates.

For example, when asked for the location of the University of Keele, the reply might be '*near to Stoke-on-Trent; halfway between Manchester and Birmingham*'. This reply would be useful in many situations, but it cannot be expressed in

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terms of numerical coordinates in a straightforward way. It is not clear exactly which locations are '*near to Stoke-on-Trent*'. Furthermore, even if we could agree on point locations for the centres of the cities of Manchester and Birmingham, computing the exact midpoint of the line joining these two centres would almost certainly yield a point several miles distant from Keele.

The management of spatial knowledge embodied in such examples is one of considerable practical importance for the development of geographical information systems (GIS). Similar examples can be produced from fields other than GIS, for example robotics and the understanding of visual scenes. These problems have led to the development of 'qualitative spatial reasoning'. A useful overview of this subject, with a substantial list of references, can be found in [Coh96].

One aspect of qualitative spatial reasoning which has received much attention concerns the description of relationships between regions – for instance they might be disjoint or meet only at their boundaries. One example of this is the region connection calculus (RCC) approach [GGC96] which develops work of Clarke [Cla81,Cla85] which itself can be traced back to Whitehead. RCC is a first order theory with a primitive relation of connection between regions. From the connection relation, other relations are derived. Stell and Worboys [SW97] have given a point-free account of models of RCC.

Another approach, also concerned with relationships between pairs of regions, is the work of Egenhofer and Franzosa. Given spatial regions A and B, with interiors A°, B°, and boundaries ∂A, ∂B respectively, we can measure how A relates to B by noting which of the intersections in the sequence $(\partial A \cap \partial B,\ A^\circ \cap B^\circ,\ \partial A \cap B^\circ,\ A^\circ \cap \partial B)$ are empty and which are non-empty. This way of describing the relationship of A to B is known as the "4-intersection model", and is studied in [EF91]. The 4-intersection model is relatively coarse. For example, knowing that $A^\circ \cap B^\circ \neq \varnothing$ tells us that A and B overlap, but there are many qualitatively distinct ways in which two regions can overlap. Finer distinctions between relationships of regions can be obtained by counting the number of connected components in the intersection of A with B, and in certain other regions determined by A and B. The investigation of this use of connected components is due to Galton [Gal98]. Another way of describing the relationship of A to B, which provides for finer distinctions than in Galton's technique, is a refinement of the 4-intersection model by Egenhofer and Franzosa [EF95].

A recent development in the study of relationships between pairs of regions considers how regions which are in some sense vague, may be related to each other. Two particular approaches are the work of Clementini and Di Felice [CF96,CF97] on regions with 'broad boundaries', and Cohn and Gotts' 'egg-yolk' representation of vague regions [CG96].

In this paper we extend the work on relationships between pairs of regions to provide a mechanism for describing a region with respect to a frame of reference. To describe the location of a region in a space, one technique is to relate it to each element in a set of regions forming a partition of the space. This technique, which is essentially a coordinate description in a qualitative fashion, has only recently begun to be investigated from a qualitative spatial reasoning perspective. An earlier paper by Bittner [Bit97] introduced some of the main ideas and made use of the RCC formalism. Another use of boundaries and partitions to represent two-dimensional space is found in the work of Lundell [Lun96]. Her work is concerned with the qualitative modelling of physical fields, so has rather different motivation from ours. However, the underlying theory appears to be closely related. Foundations of a theory of spatial partitions have been provided by Erwig and Schneider [ES97].

The main contribution of the present paper is to develop some of the theoretical foundations of the description of regions with respect to a frame of reference. Such a description of a region can be regarded as an approximation to the region. However, spatial information is not only concerned with describing regions, it also needs to deal with operations on regions. Examples of operations include unions and intersections of regions. Thus a theory of approximations to regions should give an account of approximations to operations on regions. Our work does deal with this issue in some detail, and this is the principal technical contribution of the paper. We are also able to show how our work relates to the theory of rough sets [Orł97], which provides another approach to approximation, though not in a specifically spatial context.

The structure of the paper is as follows. In Section 2 we present an extended example of the kind of qualitative location provided by our theory. In Section 3 we provide a general abstract setting for the theory. The following section develops a number of examples which help to clarify the connection to other areas, including the work on rough sets. The example in section 4.5 provides the boundary-sensitive representation of location which is the central element of the paper. Section 5 continues the abstract development by considering how approximations to elements of a set can be extended to approximations to binary operations on the set. The boundary-sensitive case is developed further in Section 6, where we consider how operations on regions can be approximated by operations on this particular kind of representation. The final section of the paper provides conclusions and suggests some directions for further work.

2. Example

Figure 1 shows a partition of the plane into six regions bounded by the solid lines. These regions are referred to as `A`, `B`, `C`, `D`, `E`, and the unbounded region external to these is `U`. Three regions which are of interest are indicated by the dotted lines. These are denoted by `p`, `q`, and `r`.

Any relationship between a single region of interest and the partition leads to a table. Let us refer to the elements of the partition as 'cells'. We can construct a table by putting 1 whenever the region completely covers the cell,

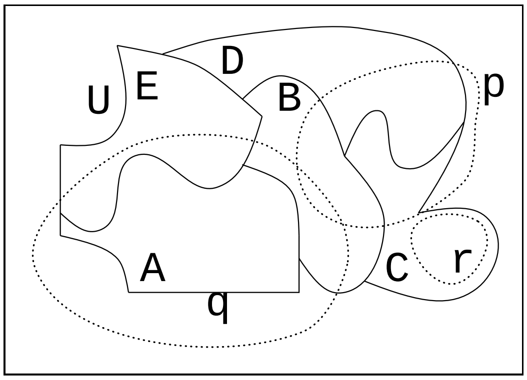


Figure 1. Plane partition with three regions of interest.

and 0 otherwise. This leads to table 1. An alternative is obtained by putting 1

Table 1
Region covers cell

	A	B	C	D	E	U
p	0	0	0	0	0	0
q	1	0	0	0	0	0
r	0	0	0	0	0	0

when the cell is not disjoint from the region and 0 otherwise. This gives table 2. In these cases the values in the table are taken from a two element set, $\{1, 0\}$. In

Table 2
Region not disjoint from cell

	A	B	C	D	E	U
p	0	1	1	1	0	1
q	1	1	0	0	1	1
r	0	0	1	0	0	0

one case these two elements can be taken as the concepts *fully covers* and *does not fully cover* and in the other as *not disjoint* and *disjoint*. If the cells in the partition corresponded to pixels, then these two possibilities would simply be two ways in which a region could be converted to a raster image. More sophisticated measures of how the regions relate to the cells arise from relationships having more than two possible outcomes. For example, writing fo for full overlap, po for partial overlap and no for no overlap, we obtain table 3. This fails to take account of the boundaries between the cells. One way to do this is to describe separately how a region relates to the interior and to the boundary of each cell.

Table 3
A three-valued relationship between regions and cells.

	A	B	C	D	E	U
p	no	po	po	po	no	po
q	fo	po	no	no	po	po
r	no	no	po	no	no	no

This produces a table in which the values are pairs. The first element in the pair describes the relationship between the region and the interior of the cell, and the second element describes the relationship between the region and the boundary of the cell. This is illustrated in table 4. For example, the entry (po, no) in the row for r and the column for C indicates that r partially overlaps the interior of C, but has no overlap with the boundary of C.

Table 4
A simple way of taking account of boundaries.

	A	B	C	D	E	U
p	(no,no)	(po,po)	(po,po)	(po,po)	(no,no)	(po,po)
q	(fo,fo)	(po,po)	(no,no)	(no,no)	(po,po)	(po,po)
r	(no,no)	(no,no)	(po,no)	(no,no)	(no,no)	(no,no)

A more detailed way of taking account of boundaries is to distinguish between parts of boundaries. To do this, we relate a region not to a single cell but to a pair consisting of the interior of a cell and the boundary shared between the cell and another cell. This gives more information than the previous way of taking account of boundaries, and leads to tables where the entries are taken from the set {fo, fbo, pbo, nbo, no}. These stand for full overlap, full boundary overlap, partial boundary overlap, no boundary overlap, and no overlap respectively. The meanings of the five relations are described in table 5.

For example, cell B shares boundaries with five other cells: A, C, D, E, and U. Rather than simply noting that the relationship of the region p to the interior of cell B and its whole boundary is (po,po), we can consider the relationship of p to each of the five pairs consisting of interior of B paired with one of the five boundaries shared between B and the other cells. We will denote the pair consisting of the interior of B and the boundary shared between B and A by (B, A), and similarly for the other pairs. The relationship of p to (B, A) is nbo, because p covers some of the interior of B, but none of the boundary shared between B and A. The relationship of p to (B, C) is pbo, because p covers some of the interior of B, and some, but not all, of the boundary shared between B and C. By considering the other parts of the boundary of B similarly, we can expand the single column headed B in table 4 into five columns as shown in table 6. By considering each of the cells in the partition in a similar fashion, we arrive at a table providing

Table 5
Five ways a region can relate to a pair of adjacent cells

Relation	Interpretation for region r and adjacent pair of cells (c, d)
fo	r fully overlaps the interior of c.
fbo	r covers all of the boundary shared between c and d, and some but not all the interior of c.
pbo	r covers only some of the boundary shared between c and d, and some but not all of the interior of c.
nbo	r covers none of the boundary shared between c and d, and some but not all of the interior of c.
no	r covers none of the interior of c.

Table 6
Producing a more detailed boundary-sensitive relationship

B		(B,A)	(B,C)	(B,D)	(B,E)	(B,U)
(po,po)	*expands to*	nbo	pbo	pbo	nbo	nbo
(po,po)		pbo	nbo	nbo	pbo	pbo
(no,no)		no	no	no	no	no

a more detailed description of how each of the regions of interest relates to the cells in the partition. The table will have one column for each ordered pair of regions which share a common boundary. For the particular example we have been studying, there are 24 columns. Space prevents us displaying the whole table, but a fragment is shown in table 7.

Table 7
Part of the boundary-sensitive relationships between regions and cells.

	(A,B)	(A,E)	(A,U)	(B,A)	(B,C)	(B,D)	(B,E)	(B,U)	(C,B)	(C,D)	(C,U)
p	no	no	no	nbo	pbo	pbo	nbo	nbo	pbo	fbo	pbo
q	fo	fo	fo	pbo	nbo	nbo	pbo	pbo	no	no	no
r	no	no	no	no	no	no	no	no	nbo	nbo	nbo

This section has discussed, in an informal way, several methods by which regions can be described with respect to a partition of the plane. These methods are discussed in more detail in a formal setting in section 4. Before doing this, we introduce, in section 3, a general framework within which all these methods can be understood.

3. Approximating Subsets

In this section we set out an abstract model which allows elements of one set to be approximated by functions to another set. The particular case of most interest to us is when the set of entities we are approximating is a set of subsets of the points of the plane.

Before outlining the details of our approach, it is useful to take a wider view of the nature of description and representation. Phenomena are described by assembling partial information obtained from a number of viewpoints. For example, it is common to describe a physical object, such as a piece of furniture by giving three numbers: its dimensions along three mutually orthogonal directions. In the relational database view of the world, things are described by attributes. These attributes have values depending on the types of the attributes. We can think of the set of attribute values for a given object as an approximation to the object.

3.1. Basic Model

These considerations lead to a simple abstract model with the following components.

- a set R of things which we want to describe.
- a set K of attributes or viewpoints from which observations can be made. K provides a framework with respect to which elements of R are described.
- a set Ω, of values for observations. We can think of Ω as the set of ways in which an element of R can relate to an element of K.
- a function $\rho : K \times R \to \Omega$. Given $r \in R$ and $k \in K$, then $\rho(k, r)$ is the result of observing r from viewpoint k. Alternatively, we can think of $\rho(k, r)$ as the way in which r relates to k.

The cases we are interested in here generally have the form that R is a set of subsets of some set P. We do not initially assume any structure on P, but for later applications it is helpful to think of P as the set of points of the plane. One possibility for R is the set of all the subsets of P. In the case that P is the set of points of a topological space, we may want to take R to be all the open sets, or some subset of these, such as the regular open sets. Further possibilities are obtained by considering closed sets, or imposing additional restrictions such as connectedness.

The function $\rho : K \times R \to \Omega$ will be referred to as the 'relationship function'. A relationship function induces another function $\alpha : R \to \Omega^K$, where Ω^K is the set of functions from K to Ω, and α is defined by taking αr to be the function $k \mapsto \rho(k, r)$. The function α is thought of as assigning to each $r \in R$ an approximation in Ω^K.

3.2. Connection with JEPD sets of relations

The idea of a set of jointly exhaustive and pairwise disjoint (JEPD) relations has been frequently used in the context of the RCC approach to qualitative spatial reasoning. It is useful to see how this notion relates to the above model.

A set B of binary relations on a set R is said to be JEPD if for every pair $(r, s) \in R \times R$ there is exactly one relation $\sim \in B$ such that $r \sim s$ is true. If B is a set of JEPD relations on R, we have a function $\rho : R \times R \to B$, taking each (r, s) to the unique relation to which it belongs. In fact, since a JEPD set of relations is exactly the same as a partition of $R \times R$, the concepts of a JEPD set of relations on R, and of a surjective function defined on $R \times R$ are essentially the same.

In many cases the set K is a subset of R. For example, R could be all subsets of a set P and K a partition of P. In a specifically spatial context, R could be all regular closed sets in the plane, and K could be a subset of these which cover the whole space P. Suppose then that K is a subset of R, and we have a JEPD set of relations on R. We obtain a representation function $\rho : K \times R \to \Omega$ by taking Ω to be the JEPD set of relations, and ρ takes (k, r) to the unique relation holding between k and r. The function ρ may not be surjective, since the nature of K may mean that some of the relations are never realized. Several of the specific examples discussed below arise from a set of JEPD relations in this way.

4. Particular Ways of Approximating Subsets

This section presents a sequence of six examples of relationship functions $\rho : K \times R \to \Omega$ which fit into the framework of the previous section. In each case there is an underlying set P, and the purpose of the relationship function is to describe certain subsets of P, that is the elements of R, by specifying how each $r \in R$ relates to each element k of the framework K. The sequence of six examples falls into two parts. In the first part, covered in sections 4.1 – 4.3, P is not assumed to have any particular structure, such as being a topological space. In the second part, covered in sections 4.4 – 4.6, we assume that the elements of P are the points of the plane with its usual topological structure. Sections 4.4, and 4.5 introduce two boundary-sensitive ways of approximating regular closed subsets of the plane. These two ways are compared in section 4.6. A summary of some features of the six examples is given in section 4.7.

4.1. Universal and Existential Relationships

The following two simple cases are basic examples from which many of the later ones can be constructed. They also play a fundamental role in the notion of rough sets. We will first present the general case in a formal way, and then give an example to illustrate the theoretical treatment.

Define $\Omega = \{1, 0\}$, and take K to be an arbitrary set of subsets of P. In this context two fundamental relationship functions can be defined:

$$\rho_\forall(k,r) = \begin{cases} 1 \text{ if } \forall a \in k \cdot a \in r \\ 0 \text{ otherwise} \end{cases} \qquad \rho_\exists(k,r) = \begin{cases} 1 \text{ if } \exists a \in k \cdot a \in r \\ 0 \text{ otherwise} \end{cases}$$

The relationships represented by these functions capture the concepts 'part of' and 'not disjoint from' respectively. That is, $\rho_\forall(k,r)$ holds if and only if k is a part of r, and $\rho_\exists(k,r)$ holds if and only if k is not disjoint from r.

To illustrate the ideas we offer the diagrams in figure 2. In these diagrams

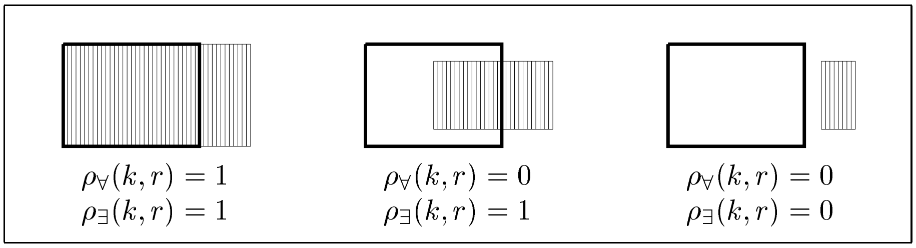


Figure 2. Three ways a region may relate to a cell.

the vertically shaded rectangle represents r, and the unshaded rectangle with thick edges represents k. In interpreting these diagrams, and several later ones, it should be noted that they are merely Venn diagrams, with no spatial content. Although k and r are drawn as regions in space, there is no assumption that that the set P or its subsets has any spatial structure.

The special case where K is a partition of the set P, and R is the set of all subsets of P, is studied in the theory of rough sets. This is illustrated in figure 3 by representing a partition of P as the division of a square into smaller squares. In this figure, K consists of sixteen small squares. We can think of the elements

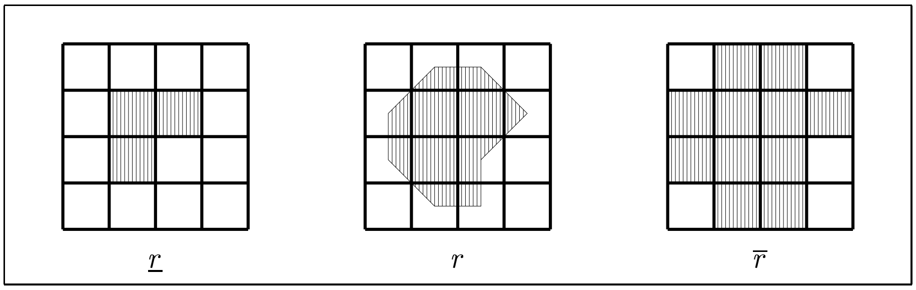


Figure 3. Two extreme approximations to r by elements of Ω^K.

of Ω^K as crude subsets of P. That is, those subsets of P which are unions of elements of K. The central diagram in figure 3 shows a vertically shaded region

r, representing a particular subset of P. The two diagrams either side, labelled $\underline{r}$ and $\overline{r}$, show the two extreme ways in which r can be approximated by a union of elements of K. The subset $\underline{r}$ consists of the elements of P lying in elements of K wholly covered by r. The subset $\overline{r}$ consists of the elements of P lying in elements of K partly or wholly covered by r.

Having seen an example, we now present a formal treatment of the general situation. Let us denote the approximation functions induced by $\rho_\forall$ and $\rho_\exists$ by $\alpha_\forall$ and $\alpha_\exists$ respectively. Each is a function from R to Ω^K. The function $\alpha_\forall$ when applied to $r \in R$ yields a function $\alpha_\forall\, r : K \to \Omega$, and similarly for $\alpha_\exists$. These two functions can be described as follows.

$$(\alpha_\forall\, r)\, k = \begin{cases} 1 \text{ if } \forall a \in k \cdot a \in r \\ 0 \text{ otherwise} \end{cases} \qquad (\alpha_\exists\, r)\, k = \begin{cases} 1 \text{ if } \exists a \in k \cdot a \in r \\ 0 \ \text{ otherwise} \end{cases}$$

Since K is a partition of P, any function $f : K \to \Omega$ gives rise to a function $f^* : P \to \Omega$, defined, for all $p \in P$, by $f^*\, p = f\, k$, where k is the unique element of K for which $p \in k$. Such an f^* has a corresponding subset of P, an element $p \in P$ being in the subset if and only if $f^*\, p = 1$. Applying this construction to the two functions $\alpha_\forall\, r$ and $\alpha_\exists\, r$, we obtain $(\alpha_\forall\, r)^*$ and $(\alpha_\exists\, r)^*$. The subsets of P corresponding to these functions are denoted $\underline{r}$ and $\overline{r}$ respectively, and will satisfy $\underline{r} \subseteq r \subseteq \overline{r}$. The subsets $\underline{r}$ and $\overline{r}$ are known in rough set theory as the lower and upper approximations to r with respect to the partition K.

4.2. Combining Existential and Universal Relationships

Provided we make the assumption that k is non-empty, there are three ways in which a subset $r \in R$ can relate to a framework element $k \in K$. These three ways are set out in table 8. Writing fo for full overlap, po for partial overlap, and

Table 8
Three ways r can relate to k.

Relationship	Formal definition
k is a part of r	$k \cap r = k$
k is partly but not fully overlapped by r	$\varnothing \subset k \cap r \subset k$
k is disjoint from r	$k \cap r = \varnothing$

no for no overlap we can define a relationship function, τ, taking values in the set $\Omega = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$.

This representation function τ arises from the combination of the previous two examples. Of the four potential combinations of results of $\rho_\forall(k, r)$ and

$\rho_\exists(k,r)$ only three can actually arise, provided we assume k non-empty. These three possibilities correspond exactly to the three cases in table 8.

$$\tau(k,r) = \begin{cases} \mathsf{fo} & \text{if } \rho_\forall(k,r) = 1 \text{ and } \rho_\exists(k,r) = 1 \\ \mathsf{po} & \text{if } \rho_\forall(k,r) = 0 \text{ and } \rho_\exists(k,r) = 1 \\ \mathsf{no} & \text{if } \rho_\forall(k,r) = 0 \text{ and } \rho_\exists(k,r) = 0 \end{cases}$$

If we identify three of the four elements of $\{1,0\} \times \{1,0\}$ with the elements of $\{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$ in the obvious way, we can define τ neatly in terms of $\rho_\forall$ and $\rho_\exists$ as $\tau\,(k,r) = (\rho_\forall\,(k,r), \rho_\exists\,(k,r))$. We can illustrate this case with the diagrams in figure 4.

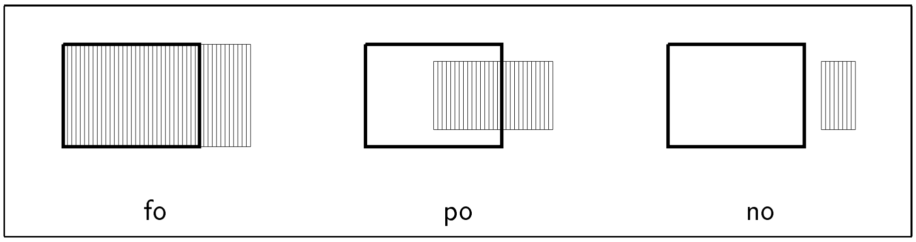


Figure 4. Examples of the three possibilities for $\tau(k,r)$.

Consider now the special case that K is a partition of the set P, and R is the set of all subsets of P. The fact that τ is obtained by combining the universal and existential relationships into one means that the elements of Ω^K are essentially the rough sets with respect to the partition K, and the approximation function $\alpha_\tau : R \to \Omega^K$ induced by τ takes each crisp subset of P to its corresponding approximation as a rough set.

In the previous section we noted that, as K is a partition of P, any function $f : K \to \Omega$ gives rise to a function $f^* : P \to \Omega$. When $\Omega = \{1,0\}$, we can identify such an f^* with a subset of P. In the case of $\Omega = \{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$, we get a rough subset of P. That is, a tripartite classification of the elements $p \in P$ into those which are definitely in the subset ($f^*\,p = \mathsf{fo}$), possibly in the subset ($f^*\,p = \mathsf{po}$), and definitely not in the subset ($f^*\,p = \mathsf{no}$). This is illustrated in figure 5. Given $r \subseteq P$, we can construct $(\alpha_\tau\,r)^* : P \to \Omega$. In figure 5, the parts of P definitely in the rough subset corresponding to $(\alpha_\tau\,r)^*$ are shaded by vertical lines, and those parts only possibly in are shaded by dots.

4.3. Cells as Pairs of Subsets

So far, we have considered cases where each framework element, or cell, $k \in K$, is a subset of P. We have not assumed any topological structure on the set P. However, a case of particular concern to us has the feature that the

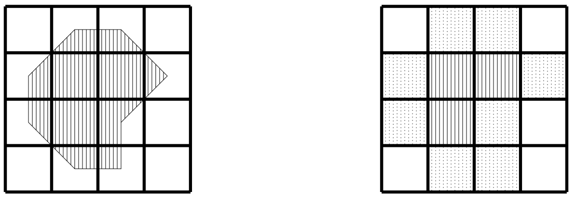


Figure 5. Rough subset approximating a crisp subset.

elements of K are closed sets in a topological space. As such, each cell $k \in K$ has two subsets of special significance: the boundary and interior.

Thus our ultimate concern is with a situation where each $k \in K$ is not a subset of P, but a pair of subsets. Ignoring, for the moment, the possibility of P carrying topological structure, it is useful to consider the simple generalization of the previous cases where K consists of pairs of subsets of P. Although this is not any more general than the previous cases, it is useful for understanding the development of the ultimate case, which appears in section 4.5.

When a cell $k \in K$ is a subset of P, we have already defined a relationship function, τ, taking values in the set $\{\mathsf{fo}, \mathsf{po}, \mathsf{no}\}$. When a cell, k, is a pair of subsets, say $k = (c, d)$, we can describe the relationship of r to k by describing how r relates to each of the pair individually. This leads to a relationship function defined by

$$\pi((c,d),\, r) = (\tau(c,r),\, \tau(d,r)).$$

This is illustrated by the diagrams in figure 6. In these diagrams, the region r is shown shaded by vertical lines; the cell (c, d) consists of d as the inner rectangular region, and c as that between the inner and outer rectangles. The reader should beware that no spatial content is implied here; the diagrams function as Venn diagrams.

4.4. Boundary-Interior pairs

We now assume a partition, F, of the plane into cells each of which is a regular closed set. The cells in F may share parts of their boundaries, but not interiors. We take R to be the set of regular closed subsets of P, denoted by $RegCl(P)$, rather than the set of all subsets of P. The set of framework elements, K, is the set F.

We can define a relationship function, β_7, making use of the previous one,

$$\beta_7\,(k, r) = \pi((k^\circ, \partial\, k), r).$$

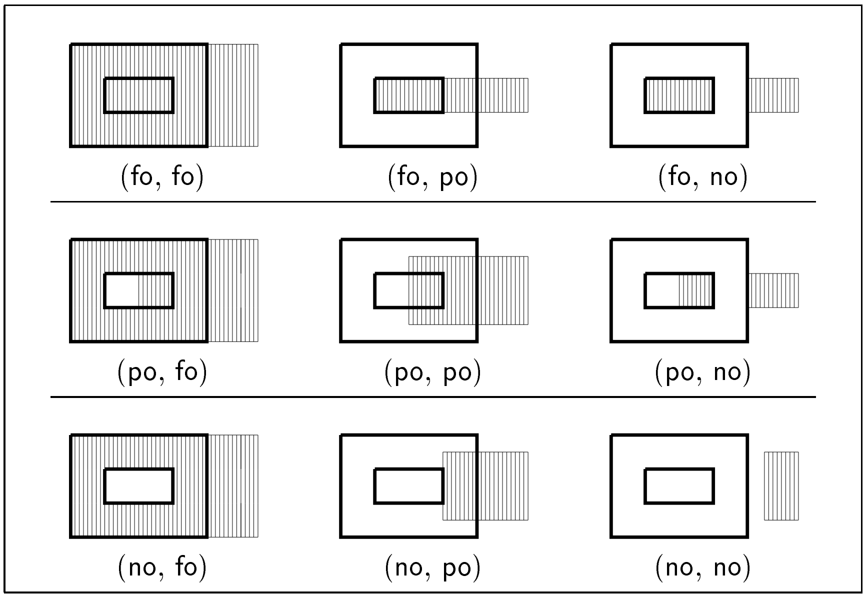


Figure 6. The possible values of $\pi((c, d), r)$.

In this definition, k° and ∂k denote the interior and the boundary respectively of k. The notation β_7 is chosen because there are seven possible values for $\beta_7(k, r)$. Of the nine possible relationships which appeared in the previous case, only seven can actually occur, due to the topological assumptions. If a regular closed set, r, wholly covers the interior of k, then it must cover all of the boundary of k. Thus the cases $(\mathsf{fo}, \mathsf{po})$, and $(\mathsf{fo}, \mathsf{no})$ never arise. We will use Ω_7 to denote the set of possible relationships:

$$\Omega_7 = \{(\mathsf{fo}, \mathsf{fo}),\ (\mathsf{po}, \mathsf{fo}),\ (\mathsf{po}, \mathsf{po}),\ (\mathsf{po}, \mathsf{no}),\ (\mathsf{no}, \mathsf{fo}),\ (\mathsf{no}, \mathsf{po}),\ (\mathsf{no}, \mathsf{no})\}.$$

4.5. Parts of Boundaries

In this section we introduce our ultimate boundary-sensitive relationship function, β_5, which provides more detailed information than β_7. We assume a partition, F, of the plane as before, and F_{adj} will denote the set of ordered pairs of F which share a common one-dimensional boundary segment. If $(c, d) \in F$ belongs to F_{adj}, we will denote the one-dimensional boundary shared between c and d by $\mathcal{B}(c, d)$. It is possible that $\mathcal{B}(c, d)$ consists of several unconnected one-dimensional pieces, but it excludes any isolated points. Thus $\mathcal{B}(c, d)$ may be strictly smaller than $c \cap \partial d$, where ∂d denotes the boundary of d.

Our set of framework elements K will be the set F_{adj}. The set, R, of things to be described will be $RegCl(P)$, the regular closed subsets of the plane, as in the previous case. The set of values, Ω has five elements, and will be denoted by Ω_5:

$$\Omega_5 = \{\mathsf{fo}, \mathsf{fbo}, \mathsf{pbo}, \mathsf{nbo}, \mathsf{no}\}.$$

The relationship function, β_5, is defined as follows, using the relationship function τ introduced in section 4.2.

$$\beta_5\left((c,d),r\right) = \begin{cases} \mathsf{fo} & \text{if } \tau(c^\circ, r) = \mathsf{fo} \\ \mathsf{fbo} & \text{if } \tau(c^\circ, r) = \mathsf{po} \text{ and } \tau(\mathcal{B}(c,d), r) = \mathsf{fo} \\ \mathsf{pbo} & \text{if } \tau(c^\circ, r) = \mathsf{po} \text{ and } \tau(\mathcal{B}(c,d), r) = \mathsf{po} \\ \mathsf{nbo} & \text{if } \tau(c^\circ, r) = \mathsf{po} \text{ and } \tau(\mathcal{B}(c,d), r) = \mathsf{no} \\ \mathsf{no} & \text{if } \tau(c^\circ, r) = \mathsf{no} \end{cases}$$

The diagrams in figure 7 illustrate the various cases. In these diagrams the cell c

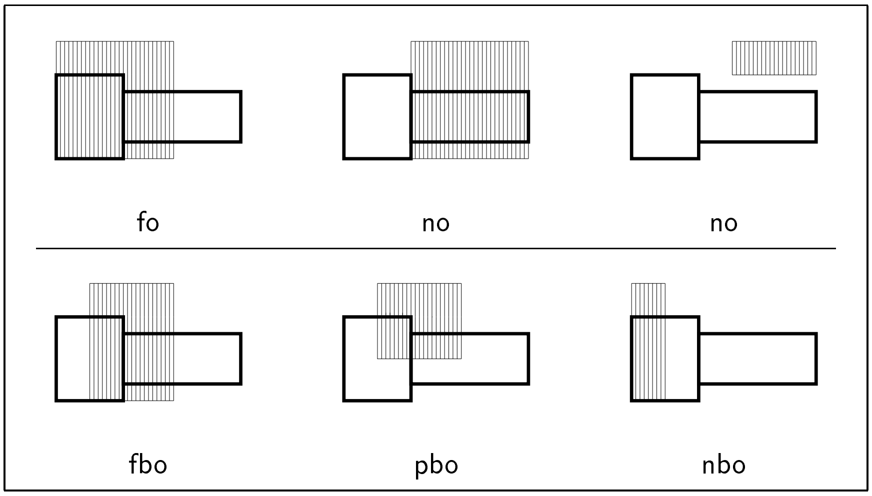


Figure 7. Possible values for $\beta_5\left((c,d),r\right)$.

is represented by the taller left-hand rectangle, and d by the right-hand rectangle. The region r is, as usual, indicated by a vertically shaded rectangle.

It is worth stressing that the relationship of r to (c,d) is primarily determined by the relationship of r to the interior of c. As the diagrams show, it is possible that $\beta_5((c,d),r) = \mathsf{no}$ even if r covers all of the boundary between c and d. Thus the value no indicates no overlap with the interior of c, but allows r to overlap the boundary of c.

4.6. Comparison of the two boundary-sensitive approaches

For a fixed partition, F, of the plane, and a given regular closed region, r, we have two boundary-sensitive relationship functions which can be used to

Table 9
Calculation of $\beta_7(k,r)$ from the values of β_5.

Condition satisfied by β_5	Value of $\beta_7(k,r)$
$\beta_5((k,b_i),r) = \mathsf{fo}$ for all i.	$(\mathsf{fo},\mathsf{fo})$
$\beta_5((k,b_i),r) = \mathsf{fbo}$ for all i.	$(\mathsf{po},\mathsf{fo})$
$\beta_5((k,b_i),r) \in \{\mathsf{fbo},\mathsf{pbo}\}$ for some i, and $\beta_5((k,b_i),r) \neq \mathsf{fbo}$ for some i.	$(\mathsf{po},\mathsf{po})$
$\beta_5((k,b_i),r) = \mathsf{nbo}$ for all i.	$(\mathsf{po},\mathsf{no})$
$\beta_5((b_i,k),r) \in \{\mathsf{fo},\mathsf{fbo}\}$ for all i, and $\beta_5((k,b_i),r) = \mathsf{no}$ for all i.	$(\mathsf{no},\mathsf{fo})$
$\beta_5((b_i,k),r) \in \{\mathsf{fo},\mathsf{fbo},\mathsf{pbo}\}$ for some i, and $\beta_5((b_i,k),r) \notin \{\mathsf{fo},\mathsf{fbo}\}$ for some i, and $\beta_5((k,b_i),r) = \mathsf{no}$ for all i.	$(\mathsf{no},\mathsf{po})$
$\beta_5((b_i,k),r) = \mathsf{no}$ for all i, and $\beta_5((k,b_i),r) = \mathsf{no}$ for all i.	$(\mathsf{no},\mathsf{no})$

describe r. The relationship function β_7 uses the framework F and takes values in Ω_7. The relationship function β_5 uses the framework F_{adj} of adjacent pairs of cells from F, and takes values in Ω_5.

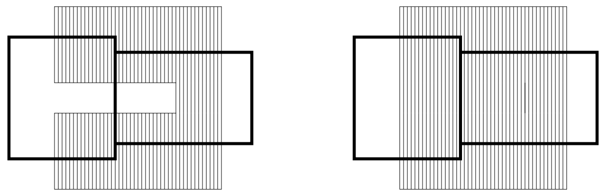
The aim of this section is to compare these two relationship functions. As we have already noted, knowing that $\beta_5((c,d),r) = \mathsf{no}$ does not tell us whether r overlaps any of the boundary shared between c and d. However, this information can always be obtained from the value of $\beta_5((d,c),r)$. In general, we will see that β_5 provides strictly more information than β_7. Given the approximation to r induced by β_5, we can calculate the approximation to r induced by β_7. However, the reverse construction is not possible: it is possible to find two regions with distinct approximations with respect to β_5, but having the same approximation with respect to β_7. Thus, the β_5 representation provides for finer distinctions than β_7.

Suppose that we have a cell $k \in F$, which is adjacent to cells $b_1, \cdots, b_n$. The value of $\beta_7(k,r)$ can be calculated from the values of β_5 as shown in table 9

An example of two distinct regions which are distinguished by β_5 but not by β_7 appears in figure 8. The region r is shown shaded, and the cell c is that on the left hand side, and cell d that on the right.

4.7. Summary

Table 10 summarizes the six cases we have considered. In each case we have a set P, and certain subsets of P constitute the set R of entities we want to approximate. Each case also has a set, K, of framework elements, or cells, a set, Ω of possible relationships, and a relationship function from $K \times R$ to Ω. In



$\beta_5((c,d),r) = \mathsf{pbo}$ $\beta_5((c,d),r) = \mathsf{fbo}$
$\beta_5((d,c),r) = \mathsf{pbo}$ $\beta_5((d,c),r) = \mathsf{fbo}$
$\beta_7(c,r) = (\mathsf{po},\mathsf{po})$ $\beta_7(c,r) = (\mathsf{po},\mathsf{po})$
$\beta_7(d,r) = (\mathsf{po},\mathsf{po})$ $\beta_7(d,r) = (\mathsf{po},\mathsf{po})$

Figure 8. Two regions distinguished by β_5, but not by β_7

the first four cases P is an arbitrary set, and R is the set of all subsets of P, denoted $\mathcal{P}P$. In the final two cases, P is assumed to be the set of points in the plane, and F is a partition of the plane into cells which may share boundaries but not interiors. F_{adj} is defined to be the set of ordered pairs of adjacent cells in F which share a common boundary. In these two cases, we are interested in approximating elements of $RegCl(P)$: the set of regular closed subsets of P.

Table 10
Summary of figure-ground relationships

Set of framework elements K	Set of regions approximated R	Set of possible relationships Ω	Relationship function $K \times R \to \Omega$
$K \subseteq \mathcal{P}P$	$\mathcal{P}P$	$\{1,0\}$	$\rho_\forall$
$K \subseteq \mathcal{P}P$	$\mathcal{P}P$	$\{1,0\}$	$\rho_\exists$
$K \subseteq \mathcal{P}P$	$\mathcal{P}P$	$\{\mathsf{fo},\mathsf{po},\mathsf{no}\}$	τ
$K \subseteq \mathcal{P}P \times \mathcal{P}P$	$\mathcal{P}P$	$\{\mathsf{fo},\mathsf{po},\mathsf{no}\} \times \{\mathsf{fo},\mathsf{po},\mathsf{no}\}$	π
$K = F$	$RegCl(P)$	$\{(\mathsf{fo},\mathsf{fo}),(\mathsf{po},\mathsf{fo}),(\mathsf{po},\mathsf{po}),(\mathsf{po},\mathsf{no}),$ $(\mathsf{no},\mathsf{fo}),(\mathsf{no},\mathsf{po}),(\mathsf{no},\mathsf{no})\}$	β_7
$K = F_{adj}$	$RegCl(P)$	$\{\mathsf{fo},\mathsf{fbo},\mathsf{pbo},\mathsf{nbo},\mathsf{no}\}$	β_5

5. Approximating Binary Operations

5.1. Introduction

So far our theory considers a set of subsets R and approximations to these by elements of Ω^K. In many cases R is not merely a set, but has algebraic structure in the form of operations on the set. One example would occur when R is a collection of subsets closed under unions and intersections. Another case would be where R is that set of all regular closed subsets of a topological space, and we consider the operations of meet and join on this set.

In this section we look at what it means for binary operations on R to be approximated by operations on Ω^K. In order to develop the theory we need to introduce a partial order on the elements of Ω. For the examples of Ω in the previous sections, the appropriate orderings are the linear ones $1 > 0$, $\mathsf{fo} > \mathsf{po} > \mathsf{no}$, and $\mathsf{fo} > \mathsf{fbo} > \mathsf{pbo} > \mathsf{nbo} > \mathsf{no}$.

A partial order on Ω leads to a partial order on Ω^K by defining for any $f, g : K \to \Omega$, $f \leq g$ if and only if for all $k \in K$, $f\,k \leq g\,k$. Note that in all of the above cases Ω is a finite chain, so is clearly complete. Since Ω is complete it follows that Ω^K is also complete.

5.2. General Theory

Suppose that X and Y are partially ordered sets (posets) and that $f : X \to Y$ is an order preserving function i.e. if $x_1 \leq x_2$ then $fx_1 \leq fx_2$. For our particular applications the reader should think of the elements of Y as being approximations to elements of X, with f the function assigning to each $x \in X$, its approximation $fx \in Y$. Now let $\star : X \times X \to X$ be an order preserving binary operation, where $X \times X$ is ordered by $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$. One might attempt to formulate the notion of a binary operation $\otimes$ on Y approximating $\star$ with respect to f by demanding that the following diagram commute.

$$\begin{tikzcd}
X \times X \arrow[r, "\star"] \arrow[d, "f \times f"'] & X \arrow[d, "f"'] \\
Y \times Y \arrow[r, "\otimes"] & Y
\end{tikzcd}$$

However, in many cases, including the ones of relevance to this paper, no operations $\otimes$ satisfy this condition. Thinking in terms of approximation this is only to be expected; the function f involves a loss of detail.

There are two ways of weakening the requirement that the above diagram commute. We can ask that for all $x_1, x_2 \in X$, $(fx_1) \otimes (fx_2) \leq f(x_1 \star x_2)$. Alternatively, we can ask that $(fx_1) \otimes (fx_2) \geq f(x_1 \star x_2)$. Let us denote by

Lower$(\star, f)$ the set of all order preserving binary operations, $\otimes$, on Y which satisfy the first of these inequalities. We will use *Upper*$(\star, f)$ to denote those satisfying the second inequality.

In general *Lower*$(\star, f)$ will contain many operations which cannot be regarded as useful approximations to $\star$. For example, if Y has a least element, $\bot$, then the operation defined by $y_1 \otimes y_2 = \bot$ belongs to *Lower*$(\star, f)$. The set *Lower*$(\star, f)$ is partially ordered by putting $\otimes_1 \leq \otimes_2$ if for all $y_1, y_2 \in Y$, $y_1 \otimes_1 y_2 \leq y_1 \otimes_2 y_2$. This ordering captures the idea that $\otimes_2$ is a better approximation to $\star$ than $\otimes_1$. So, if *Lower*$(\star, f)$ has a greatest element, then this is the best approximation to $\star$ among the operations in *Lower*$(\star, f)$. Similarly, if *Upper*$(\star, f)$ has a least element, then this will be the best approximation to $\star$ among the elements in *Upper*$(\star, f)$.

The following theorem shows that provided Y is complete i.e. has all meets and joins, these two best approximations always exist. In this case any operation $\star$ can be assigned a pair of operations on Y which taken together provide the best approximation to $\star$ with respect to f.

Theorem 1. If Y has all meets then *Lower*$(\star, f)$ has a greatest element, and if Y has all joins then *Upper*$(\star, f)$ has a least element.

Proof. Suppose that Y has all meets, and for any $y_1, y_2 \in Y$ define

$$U(y_1, y_2) = \{f(a \star b) \in Y \mid fa \geq y_1 \text{ and } fb \geq y_2\}.$$

Now define the operation $\star_{min}$ on Y by taking $y_1 \star_{min} y_2$ to be the meet of the set $U(y_1, y_2)$. It is straightforward to verify that $\star_{min}$ is the greatest element of *Lower*$(\star, f)$.

To construct the least element of *Upper*$(\star, f)$, define

$$L(y_1, y_2) = \{f(a \star b) \in Y \mid fa \leq y_1 \text{ and } fb \leq y_2\}.$$

The operation $\star_{max}$ defined by taking $y_1 \star_{max} y_2$ to be the join of the set $L(y_1, y_2)$. □

In fact the constructions of $\star_{min}$ and $\star_{max}$ are instances of constructions in category theory known as right and left Kan extensions. The above theorem is simply a special case of a well-known and much more general construction [Bor94, p123].

6. Operations on Boundary-Sensitive Approximations

This section develops the boundary-sensitive relationship model which was introduced in section 4.5.

Recall that we have a partition of the plane, and that our framework K consists of pairs of cells of the partition which share a common boundary segment.

We are interested in describing arbitrary regular closed subsets of the plane with respect to this framework. The relationship function $\beta_5 : K \times R \to \Omega$ allows us to define the approximation function $\alpha : R \to \Omega^K$. The aim of this section is to provide an explicit description of the approximations to the binary operations of meet and join in R by operations on Ω^K.

6.1. Approximating Operations: Simple Case

Before considering the general case of the operations on Ω^K induced by the approximation function α, we deal with a simpler case.

Given a particular element of our framework $(c, d) \in K$, we can approximate $r \in R$ by $(\alpha\, r)\,(c, d)$. That is we can simply consider how r relates to a single framework element (c, d), rather than to all framework elements simultaneously. Thus for each (c, d) we have a function $\gamma\,(c, d) : R \to \Omega$.

Now, the set R is well known to be a Boolean algebra, and we will write $\wedge$, and $\vee$ for the meet and join operations to distinguish them from the intersection and union operations in the Boolean algebra of all subsets of points in the plane. We can apply the theory of section 5 to the particular case of R approximated by Ω with respect to the function $\gamma\,(c, d)$.

From the operation $\wedge$ on R we obtain the operations $\wedge_{min}$ and $\wedge_{max}$ on Ω. By using the construction in proof of Theorem 1, some straightforward calculations lead to the following explicit description of these operations.

$\wedge_{min}$	no	nbo	pbo	fbo	fo
no	no	no	no	no	no
nbo	no	no	no	no	nbo
pbo	no	no	no	pbo	pbo
fbo	no	no	pbo	fbo	fbo
fo	no	nbo	pbo	fbo	fo

$\wedge_{max}$	no	nbo	pbo	fbo	fo
no	no	no	no	no	no
nbo	no	nbo	nbo	nbo	nbo
pbo	no	nbo	pbo	pbo	pbo
fbo	no	nbo	pbo	fbo	fbo
fo	no	nbo	pbo	fbo	fo

Similarly, the operations $\vee_{min}$ and $\vee_{max}$ on Ω obtained from the operation $\vee$ on R can be described explicitly as follows.

$\vee_{min}$	no	nbo	pbo	fbo	fo
no	no	nbo	pbo	fbo	fo
nbo	nbo	nbo	pbo	fbo	fo
pbo	pbo	pbo	pbo	fbo	fo
fbo	fbo	fbo	fbo	fbo	fo
fo	fo	fo	fo	fo	fo

$\vee_{max}$	no	nbo	pbo	fbo	fo
no	no	nbo	pbo	fbo	fo
nbo	nbo	nbo	pbo	fo	fo
pbo	pbo	pbo	fo	fo	fo
fbo	fbo	fo	fo	fo	fo
fo	fo	fo	fo	fo	fo

6.2. Approximating Operations: General Case

We have introduced the set of functions from K to Ω as approximations to the elements of R. For each region $r \in R$, its approximation is obtained by applying the function α. Now we consider how the operations on R can be approximated by operations on Ω^K. Using the general theory developed in section 5 above, any binary operation $\star$ on R gives rise to two binary operations on Ω^K as in the diagram.

$$\begin{tikzcd} R \times R \arrow[r, "\star"] \arrow[d, "\alpha \times \alpha"'] & R \arrow[d, "\alpha"] \\ \Omega^K \times \Omega^K \arrow[r, shift left, "\star_{min}"] \arrow[r, shift right, "\star_{max}"'] & \Omega^K \end{tikzcd}$$

Although the general theory referred to gives us a construction for the operations on Ω^K, it is necessary to do some further work in order to obtain an explicit description of the operations which does not refer to the function α. Before presenting this explicit description, it is useful to consider a straightforward way of constructing binary operations on Ω^K from binary operations on Ω.

We can extend any binary operation, $\star$, on Ω to a binary operation on Ω^K. For functions $f, g \in \Omega^K$, define $f \star g$ by

$$(f \star g)(c_i, c_j) = (f\,(c_i, c_j)) \star (g\,(c_i, c_j)).$$

When $\star$ on Ω is $\wedge_{max}$ or $\vee_{min}$, as defined in section 6.1, this technique provides the same corresponding operations on Ω^K as we get from applying the construction in theorem 1 to the operations $\wedge$ and $\vee$ in R.

However, in the case of $\wedge_{min}$ and $\vee_{max}$ the situation is somewhat more complicated. To see why this is, consider a cell c_1 sharing a boundary with both c_2 and c_3. Suppose f and g are functions with the following properties.

$$\begin{array}{ll} f(c_1, c_2) = \mathsf{nbo} & g(c_1, c_2) = \mathsf{pbo} \\ f(c_1, c_3) = \mathsf{pbo} & g(c_1, c_3) = \mathsf{fbo} \end{array}$$

We can see that the regions represented by f and by g must have a non-empty intersection including part of the boundary between c_1 and c_3. However this fact is not apparent if we only consider the values of $f(c_1, c_2)$ and $g(c_1, c_2)$. This means that the value of $(f \wedge_{min} g)(c_1, c_2)$ is not determined solely by the values of f and g on (c_1, c_2). It is necessary to consider the values of f and g on all pairs of the form $(c_1, c_k) \in K$. That is, the set

$$\{(f\,(c_1, c_k),\ g\,(c_1, c_k)) \in \Omega \times \Omega \mid (c_1, c_k) \in K\}.$$

We can think of this set as being a kind of neighbourhood of the partition element c_1. Such a neighbourhood can be constructed for each partition element c_i, and we define

$$N(i) = \{(f\,(c_i, c_k),\ g\,(c_i, c_k)) \in \Omega \times \Omega \mid (c_i, c_k) \in K\}.$$

This set is needed for the explicit descriptions of the operations $\vee_{max}$ and $\wedge_{min}$ on Ω^K. We will deal with the case of $\wedge_{min}$ first. Define the set

$$\Phi = \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid \omega_1 \wedge \omega_2 \geq \mathsf{pbo},\ \ \omega_1 \vee \omega_2 \geq \mathsf{fbo}\}.$$

Now for any $X \subseteq \Omega \times \Omega$ we define a binary operation $\triangle_X$ by

$\triangle_X$	no	nbo	pbo	fbo	fo
no	no	no	no	no	no
nbo	no	$\varphi(X)$	$\varphi(X)$	$\varphi(X)$	nbo
pbo	no	$\varphi(X)$	$\varphi(X)$	pbo	pbo
fbo	no	$\varphi(X)$	pbo	fbo	fbo
fo	no	nbo	pbo	fbo	fo

where $\varphi(X) = \begin{cases} \mathsf{no} & \text{if } X \cap \Phi = \varnothing \\ \mathsf{nbo} & \text{if } X \cap \Phi \neq \varnothing \end{cases}$

By applying this to a particular case of X we obtain the explicit description of $\wedge_{min}$ on Ω^K.

Theorem 2. For any $f, g \in \Omega^K$ and for any $(c_i, c_j) \in K$,

$$(f \wedge_{min} g)\,(c_i, c_j) = (f\,(c_i, c_j)) \triangle_{N(i)} (g\,(c_i, c_j)),$$
$$(f \wedge_{max} g)\,(c_i, c_j) = (f\,(c_i, c_j)) \wedge_{max} (g\,(c_i, c_j)).$$

□

The case of $\vee_{max}$ is similar. The set Ψ is defined by

$$\Psi = \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid \omega_1 \wedge \omega_2 \leq \mathsf{nbo},\ \ \omega_1 \vee \omega_2 \leq \mathsf{pbo}\},$$

and for any $X \subseteq \Omega \times \Omega$ we define $\bigtriangledown_X$ by

$\bigtriangledown_X$	no	nbo	pbo	fbo	fo
no	no	nbo	pbo	fbo	fo
nbo	nbo	nbo	pbo	$\psi(X)$	fo
pbo	pbo	pbo	$\psi(X)$	$\psi(X)$	fo
fbo	fbo	$\psi(X)$	$\psi(X)$	$\psi(X)$	fo
fo	fo	fo	fo	fo	fo

where $\psi(X) = \begin{cases} \mathsf{fo} & \text{if } X \cap \Psi = \varnothing \\ \mathsf{fbo} & \text{if } X \cap \Psi \neq \varnothing \end{cases}$

Theorem 3. For any $f, g \in \Omega^K$ and for any $(c_i, c_j) \in K$,

$$(f \vee_{min} g)\,(c_i, c_j) = (f\,(c_i, c_j)) \vee_{min} (g\,(c_i, c_j)),$$
$$(f \vee_{max} g)\,(c_i, c_j) = (f\,(c_i, c_j)) \bigtriangledown_{N(i)} (g\,(c_i, c_j)).$$

□

7. Conclusions and Further Work

This paper has presented a formal model of qualitative location which takes account of how a region relates to the boundaries of cells in a partition of the plane. We have shown how this approach extends the basic rough-set concept, in which the boundaries of cells have no special status. The principal novel ingredients in our work are the facts that our model takes account of boundaries, and that we have provided a detailed theory of how operations on regions are related to operations on their qualitative approximations.

There are several directions for further work. We intend to extend the theory to deal with boundaries in a more general sense. The current theory starts from a partition of the ground into cells which may share boundaries but do not actually overlap. That is, the points in the interiors of the cells are subject to an equivalence relation, the equivalence classes of which are the interiors of the cells in the partition. The equivalence relation models a notion of indistinguishability between points. Smyth [Smy92, p660], notes that several authors, going back at least to Poincaré, have argued that indistinguishability should be reflexive and symmetric, but not necessarily transitive.

If the relation on points in the interiors of cells is reflexive and symmetric, but not necessarily transitive, then every point lies in at least one cell, rather than in exactly one cell as would be the case with a partition. This corresponds to a generalization of the notion of partition in which the cells are permitted to overlap. It is clear that our figure ground relations can be generalized to this more general context. Now, however our boundaries are not necessarily one dimensional but can be regions themselves. This work needs further development, but one specific question would be to determine how these more general figure-ground relations are related to the work by Clementini and Di Felice on broad boundaries [CF97].

Another area for further work is to consider not just one partition, but many, and to develop a theory of how approximations to regions change under changes in the underlying partition. This would have particular application to cartographic generalization. The term 'generalization' in the context of geographical maps means the transformation of data at one level of detail to data at a lower, i.e. more coarse, level of detail. Some work on a formal basis for generalization has been carried out by Stell and Worboys [SW98]. The techniques of the present paper can be investigated in this context. If K and H are frameworks, where K is coarser than H, it should be possible to define a corresponding generalization function from Ω^H to Ω^K.

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