Possible Definitions of an 'A Priori' Granule in General Rough Set Theory

A. Mani

Member, Calcutta Mathematical Society
E-Mail: *a.mani@member.ams.org*
Homepage: http://www.logicamani.co.cc

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ABSTRACT

We introduce an abstract framework for general rough set theory from a mereological perspective and consider possible concepts of 'a priori' granules and granulation in the same. The framework is ideal for relaxing many of the relatively superfluous set-theoretic axioms and for improving the semantics of many relation based, cover-based and dialectical rough set theories. This is a relatively simplified presentation of a section in three different recent research papers by the present author.

Outline

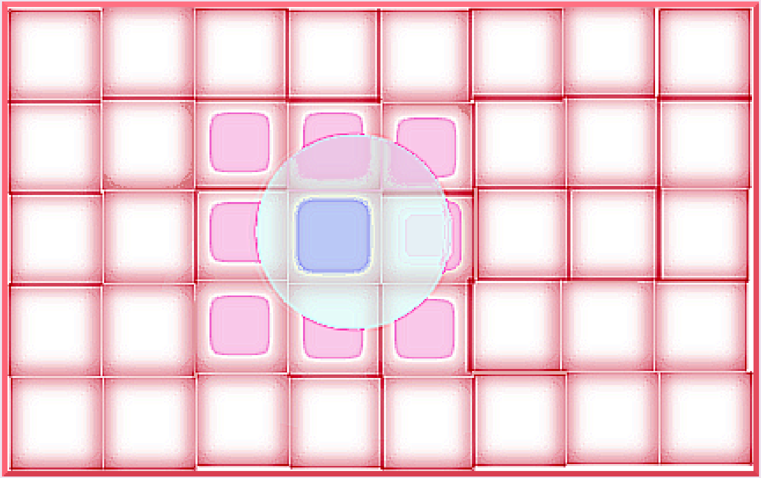
INTRODUCTION

- A Space of interest in General RST consists of objects, some of which are more definite than others and some others are more approximate than others.
- In general we use operations on the objects to form their approximations and may be able to define the concept of definiteness in terms of formulas constructed from them.
- In many situations, we may be actually able to derive this 'space of interest' from more basic objects called 'granules' possesing different properties.

Simplest Case

- **Approximation Space**: $S = \langle \underline{S}, R \rangle$, where $\underline{S}$ is a set and R is an equivalence.
- If $A \subset S$, $A^l = \bigcup\{[x] ; [x] \subseteq A\}$ and $A^u = \bigcup\{[x] ; [x] \cap A \neq \emptyset\}$ are the lower and upper approximation of A respectively
- A is *definite* iff $A^l = A = A^u$.
- Rough Incusion: $A \sqsubseteq B$ iff $A^l \subseteq B^l$ and $A^u \subseteq B^u$.
- Rough Equality: $A \approx B$ iff $A \sqsubseteq B$ and $B \sqsubseteq A$.

Example



Standard Example

Granules in RST

- Classical RST: Granules are the minimal definite elements. The set of all granules forms a granulation for the semantics.
- Classical RST: Granules are precisely the equivalence classes determined by the equivalence relation (An a priori Definition)
- In most cover based and other relation based RSTs, the most appropriate concept of granules emerge after the formulation of the semantics.
- This is problematic for semantics and applications that require some specific conditions on the granules used.
- The Lesniewski ontology based mereological approach makes heavy use of membership functions in basic definitions and has related concepts of granularity - this is problematic from the point of view of foundations.
- The scope of definitions of granules in the literature have been very restricted for many reasons.

AIAU Systems

Let S be a set and $\mathcal{K} = \{K_i\}_1^n$ be a collection of subsets of it such that $\bigcup \mathcal{K} = S$. If $X \subseteq S$, then consider the sets (with $K_0 = \emptyset$, $K_{n+1} = S$)

1. $X^{l1} = \bigcup\{K_i : K_i \subseteq X, i \in \{0, 1, ..., n\}\}$
2. $X^{l2} = \bigcup\{\cap(S \setminus K_i) : \cap_I(S \setminus K_i) \subseteq X, I \subseteq \{1, ..., n+1\}\}$
3. $X^{u1} = \bigcap\{\cup K_i : X, \subseteq \cup K_i \, i \in \{1, ..., n + 1\}\}$
4. $X^{u2} = \bigcap\{S \setminus K_i : X, \subseteq S \setminus K_i \, i \in \{0, ..., n\}\}$

The pair (X^{l1}, X^{u1}) is called a *AU-rough set* by union, while (X^{l2}, X^{u2}) a *AI-rough set* by intersection. In [AM960], we show that the elements of $\mathcal{K}$ are not the best possible granules for the approximations. Obviously the elements of $\mathcal{K}$ are not 'definite' in many senses in general.

Rough Y-Systems (RYS)

- $S = \langle S, W, \mathbf{P}, (l_i)_1^n, (u_i)_1^n, +, \cdot, \sim, 1\rangle$
- $(\forall x)\mathbf{P}xx$; $(\forall x, y)(\mathbf{P}xy, \mathbf{P}yx \longrightarrow x = y)$
- For each i, j, l_i, u_j are surjective functions : $S \longmapsto W$
- For each i, $(\forall x, y)(\mathbf{P}xy \longrightarrow \mathbf{P}(l_i x)(l_i y), \mathbf{P}(u_i x)(u_i y))$
- For each i, $(\forall x)\, \mathbf{P}(l_i x)x, \mathbf{P}(x)(u_i x)), \mathbf{P}(l_i x)(u_i l_i x), \mathbf{P}(l_i u_i x)(u_i x)$
- For each i, $(\forall x)(\mathbf{P}(u_i x)(l_i x) \longrightarrow x = l_i x = u_i x)$

In the definition of a RYS, it makes sense to relax the surjectivity of u_i, l_i. The resulting structure will be called a *general RYS*.

Continued

Overlap: $\mathbf{O}xy$ iff $(\exists z)\, \mathbf{P}zx \wedge \mathbf{P}zy$; **Underlap:** $\mathbf{U}xy$ iff $(\exists z)\, \mathbf{P}xz \wedge \mathbf{P}yz$

Proper Part: $\mathbb{P}xy$ iff $\mathbf{P}xy \wedge \neg \mathbf{P}yx$; **Overcross:** $\mathbb{X}xy$ iff $\mathbf{O}xy \wedge \neg \mathbf{P}xy$

Proper Overlap: $\mathbb{O}xy$ iff $\mathbb{X}xy \wedge \mathbb{X}yx$

Sum: $x + y = \iota z(\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wx \vee \mathbf{O}wy))$

Product: $x \cdot y = \iota z(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \mathbf{P}wy))$

Difference: $x - y = \iota z(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \neg \mathbf{O}wy))$

Associativity: Assumption: $+, \cdot$ are associative

Clarifications:

- The 'parthood relation' **P** is intended as a general form of 'rough inclusion'.
- Interestingly many semantics of general RST do not make use of any 'rough inclusions' at all as their intent is not to describe 'roughly equivalent objects'. Example: Cattaneo's BZ-algebras and variants.
- In classical RST, 'supplementation' in the stricter sense, $(\neg\mathbf{P}xy \longrightarrow \exists z(\mathbf{P}zx \wedge \neg\mathbb{O}zy))$ does not hold), while the weaker version $(\neg\mathbf{P}xy \longrightarrow \exists z(\mathbf{P}zx \wedge \neg\mathbf{O}zy))$ is trivially satisfied due to the existence of the empty object ($\emptyset$).
- Non-transitivity of **P** can be the result of adding attributes in even simpler cases.

Continued

- Handle-Door-House example situations can happen in a more general sense.
- Problem of Ontological Innocence
- Sum operation can cause 'Plural reference', but is not a major problem here as we are dealing with an abstract object.
- General RYS with transitivity of the parthood relation can be related to 'property systems' of Vakarelov and include 'information quantum relation system' (Theorem)

Granules: Possible Properties

Representability, RA $\forall i, (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\ y_1 + y_2 + \ldots + y_r = x^{l_i}$ and
$(\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\ y_1 + y_2 + \ldots + y_p = x^{u_i}$

Weak RA, WRA $\forall i, (\forall x \exists y_1, \ldots y_r \in \mathcal{G})\ t_i(y_1, y_2, \ldots y_r) = x^{l_i}$ and
$(\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\ t_i(y_1, y_2, \ldots y_p) = x^{u_i}$

Absolute Crispness, ACG For each i, $(\forall y \in \mathcal{G})\ y^{l_i} = y^{u_i} = y$

Weak Crispness, WCG $\exists i, (\forall y \in \mathcal{G}) y^{l_i} = y^{u_i} = y$.

Mereological Atomicity,MER $\exists i$,
$(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy,\ x^{l_i} = x^{u_i} = x \longrightarrow x = y)$

Granules: Continued

Lower Stability,LS $\exists i, (\forall y \in \mathcal{G})(\forall x \in S)\,(\mathbf{P}yx \longrightarrow \mathbf{P}(y)(x^{l_i}))$

Upper Stability,US $\exists i, (\forall y \in \mathcal{G})(\forall x \in S)\,(\mathbf{O}yx \longrightarrow P(y)(x^{u_i}))$

Stability,ST Shall be the same as the satisfaction of LS and US.

Absolute Stability,AS Same as the satisfaction of ST for every i

No Overlap,NO $(\forall x, y \in \mathcal{G})\neg\mathbb{O}xy$

Full Underlap, FU $\exists i, (\forall x, y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z^{u_i} = z$

Unique Underlap,UU For at least one i, $(\forall x, y \in \mathcal{G})(\mathbb{P}xz, \mathbb{P}yz, z^{l_i} = z^{u_i} = z, \mathbb{P}xb, \mathbb{P}yb, b^{l_i} = b^{u_i} = b \longrightarrow z = b)$

Definition

Definition

A subset $\mathcal{G}$ of S in a RYS will be said to be an *admissible set of granules* provided the properties WRA, LS and FU are satisfied by it. Using more properties we can define posets of possible granulations

- $\mathcal{K}$ in *AUAI* systems are admissible granulations. But these can be refined (see AM960 for details) for the same approximations. So I refer to the former as 'initial granules' and the latter are relatively *refined granules*.
- Other definitions of granules exist in the literature, but they do not refer to specific properties or are not general enough

Conclusion : We have axiomatically defined a Poset of granules from a mereological perspective of general rough set theory. The motivation has been in the requirements of foundational studies, applications to cover based RST and dialectical RST.

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