Summary of the Mathematical Notation

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Bucharest DEPLOY 2-day course, 14-16/07/10, ETH Zurich

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Swiss Federal Institute of Technology Zurich

Outline

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Foundation for Deductive and Formal Proofs

- Reason: We want to understand how proofs can be mechanized.
- Topics:
  - Concepts of Sequent and Inference Rule.
  - Backward and Forward reasoning
  - Basic Inference Rules.

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Sequent

- Sequent is the generic name for "something we want to prove"
- We shall be more precise later

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Inference Rule

- An inference rule is a tool to perform a formal proof
- It is denoted by:

$$\frac{A}{C}$$

- A is a (possibly empty) collection of sequents: the antecedents
- C is a sequent: the consequent

The proofs of each sequent of A
——— together give you ———
a proof of sequent C

Backward and Forward Reasoning

Given an inference rule $\frac{A}{C}$ with antecedents A and consequent C

- Forward reasoning: $\frac{A}{C}\downarrow$
  Proofs of each sequent in A give you a proof of the consequent C
- Backward reasoning: $\frac{A}{C}\uparrow$
  In order to get a proof of C, it is sufficient to have proofs of each sequent in A

Proofs are usually done using backward reasoning

"Executing" the Proof of a Sequent S (backward reasoning)

We are given:

- a collection $\mathcal{T}$ of inference rules of the form $\frac{A}{C}$
- a sequent container K, containing S initially

while K is not empty

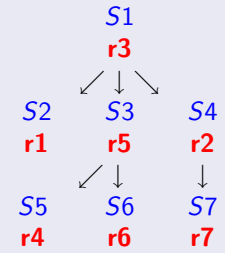
choose a rule $\frac{A}{C}$ in $\mathcal{T}$ whose consequent C is in K;

replace C in K by the antecedents A (if any)

This proof method is said to be goal oriented.

Proof of $S1$

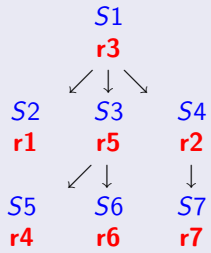
$\mathbf{r1}\frac{}{S2}$ $\mathbf{r2}\frac{S7}{S4}$ $\mathbf{r3}\frac{S2\ \ S3\ \ S4}{S1}$ $\mathbf{r4}\frac{}{S5}$ $\mathbf{r5}\frac{S5\ \ S6}{S3}$ $\mathbf{r6}\frac{}{S6}$ $\mathbf{r7}\frac{}{S7}$



- The proof is a tree
- We have shown here a depth-first strategy

Alternate Representation of the Proof Tree

A vertical representation of the proof tree:



S1	r3
S2	r1
S3	r5
S5	r4
S6	r6
S4	r2
S7	r7

Proof of $S1$

$$\mathbf{r1}\,\frac{}{S2} \qquad \mathbf{r2}\,\frac{S7}{S4} \qquad \mathbf{r3}\,\frac{S2 \;\; S3 \;\; S4}{S1} \qquad \mathbf{r4}\,\frac{}{S5} \qquad \mathbf{r5}\,\frac{S5 \;\; S6}{S3} \qquad \mathbf{r6}\,\frac{}{S6} \qquad \mathbf{r7}\,\frac{}{S7}$$

S1	r3
S2	r1
S3	r5
S5	r4
S6	r6
S4	r2
S7	r7

More on Sequent

- We supposedly have a Predicate Language (not defined yet)
- A sequent is denoted by:

$$\mathrm{H} \vdash G$$

- H is a (possibly empty) collection of predicates: the hypotheses
- G is a predicate: the goal

Meaning ...

Under the hypotheses of collection H, prove the goal G

Basic Inference Rules of Mathematical Reasoning

- HYPOTHESIS: If the goal belongs to the hypotheses of a sequent, then the sequent is proved,
- MONOTONICITY: Once a sequent is proved, any sequent with the same goal and more hypotheses is also proved,
- CUT: If you succeed in proving P under H, then P can be added to the collection H for proving a goal G.

Basic Inference Rules

$$\frac{}{H,\ P \vdash P}\ \text{HYP} \qquad \frac{H \vdash Q}{H,\ P \vdash Q}\ \text{MON}$$

$$\frac{H \vdash P \qquad H,\ P \vdash Q}{H \vdash Q}\ \text{CUT}$$

Basic Constructs of Propositional Calculus

Given predicates P and Q, we can construct:

- CONJUNCTION: $P \wedge Q$
- IMPLICATION: $P \Rightarrow Q$
- NEGATION: $\neg P$

Syntax

$$\begin{array}{rcl} \mathit{Predicate} & ::= & \mathit{Predicate} \wedge \mathit{Predicate} \\ & & \mathit{Predicate} \Rightarrow \mathit{Predicate} \\ & & \neg\ \mathit{Predicate} \end{array}$$

- This syntax is ambiguous.

More on Syntax

- Pairs of matching parentheses can be added freely.
- Operator $\wedge$ is associative.
- Operator $\Rightarrow$ is not associative: $P \Rightarrow Q \Rightarrow R$ is not allowed.
- Write explicitly $(P \Rightarrow Q) \Rightarrow R$ or $P \Rightarrow (Q \Rightarrow R)$.
- Operators have precedence in this decreasing order: $\neg$, $\wedge$, $\Rightarrow$.

Extensions: Truth, Falsity, Disjunction and Equivalence

- TRUTH: $\top$
- FALSITY: $\bot$
- DISJUNCTION: $P \vee Q$
- EQUIVALENCE: $P \Leftrightarrow Q$

Syntax

$$
\begin{array}{lll}
\mathit{Predicate} & ::= & \mathit{Predicate} \wedge \mathit{Predicate} \\
 & & \mathit{Predicate} \Rightarrow \mathit{Predicate} \\
 & & \neg\ \mathit{Predicate} \\
 & & \bot \\
 & & \top \\
 & & \mathit{Predicate} \vee \mathit{Predicate} \\
 & & \mathit{Predicate} \Leftrightarrow \mathit{Predicate}
\end{array}
$$

More on Syntax

- Pairs of matching parentheses can be added freely.
- Operators $\wedge$ and $\vee$ are associative.
- Operator $\Rightarrow$ and $\Leftrightarrow$ are not associative.
- Precedence decreasing order: $\neg$, $\wedge$ and $\vee$, $\Rightarrow$ and $\Leftrightarrow$.

More on Syntax (cont'd)

- The mixing of $\wedge$ and $\vee$ without parentheses is not allowed.
- You have to write either $P \wedge (Q \vee R)$ or $(P \wedge Q) \vee R$
- The mixing of $\Rightarrow$ and $\Leftrightarrow$ without parentheses is not allowed.
- You have to write either $P \Rightarrow (Q \Leftrightarrow R)$ or $(P \Rightarrow Q) \Leftrightarrow R$

Propositional Calculus Rules of Inference (1)

- Rules about conjunction

$$\frac{\mathbf{H}, \mathbf{P}, \mathbf{Q} \vdash \mathbf{R}}{\mathbf{H}, \mathbf{P} \wedge \mathbf{Q} \vdash \mathbf{R}}\ \text{AND\_L} \qquad \frac{\mathbf{H} \vdash \mathbf{P} \quad \mathbf{H} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \wedge \mathbf{Q}}\ \text{AND\_R}$$

- Rules about implication

$$\frac{\mathbf{H}, \mathbf{P}, \mathbf{Q} \vdash \mathbf{R}}{\mathbf{H}, \mathbf{P}, \mathbf{P} \Rightarrow \mathbf{Q} \vdash \mathbf{R}}\ \text{IMP\_L} \qquad \frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \Rightarrow \mathbf{Q}}\ \text{IMP\_R}$$

Note

Rules with a double horizontal line can be applied in both directions.

Propositional Calculus Rules of Inference (2)

- Rules about disjunction

$$\frac{\mathbf{H}, \mathbf{P} \vdash \mathbf{R} \quad \mathbf{H}, \mathbf{Q} \vdash \mathbf{R}}{\mathbf{H}, \mathbf{P} \vee \mathbf{Q} \vdash \mathbf{R}}\ \text{OR\_L}$$

$$\frac{\mathbf{H}, \neg P \vdash \mathbf{Q}}{\mathbf{H} \vdash \mathbf{P} \vee \mathbf{Q}}\ \text{OR\_R}$$

Propositional Calculus Rules of Inference (3)

- Rules about negation

$$\frac{\mathbf{H}, \neg \mathbf{Q} \vdash \mathbf{P}}{\mathbf{H}, \neg \mathbf{P} \vdash \mathbf{Q}}\ \text{NOT\_L} \qquad \frac{\mathbf{H}, \mathbf{P} \vdash \bot}{\mathbf{H} \vdash \neg \mathbf{P}}\ \text{NOT\_R}$$

$$\frac{}{\mathbf{H}, \bot \vdash \mathbf{P}}\ \text{FALSE\_L}$$

$$\frac{\mathbf{H} \vdash \mathbf{P} \quad \mathbf{H} \vdash \neg \mathbf{P}}{\mathbf{H} \vdash \bot}\ \text{FALSE\_R}$$

Propositional Calculus Rules of Inference (4)

- Deriving rules:

$$\frac{\mathrm{H}, Q \vdash P \quad \mathrm{H}, \neg Q \vdash P}{\mathrm{H} \vdash P}\ \textbf{CASE}$$

$$\frac{\mathrm{H}. \neg Q \vdash \neg P}{\mathrm{H}, P \vdash Q}\ \textbf{CT\_L} \qquad \frac{\mathrm{H}. \neg P \vdash \bot}{\mathrm{H} \vdash P}\ \textbf{CT\_R}$$

$$\frac{\mathrm{H} \vdash P}{\mathrm{H} \vdash P \vee Q}\ \textbf{OR\_R1} \qquad \frac{\mathrm{H} \vdash Q}{\mathrm{H} \vdash P \vee Q}\ \textbf{OR\_R2}$$

Propositional Calculus Rules of Inference (4)

- Rewriting rules:

Predicate	Rewritten
$\top$	$\neg \bot$
$P \Leftrightarrow Q$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

- More derived rules:

$$\frac{}{H \vdash \top}\ \textbf{TRUE\_R} \qquad \frac{H \vdash P}{H, \top \vdash P}\ \textbf{TRUE\_L}$$

CLASSICAL RESULTS (1)

commutativity	$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$ $(P \Leftrightarrow Q) \Leftrightarrow (Q \Leftrightarrow P)$
associativity	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ $((P \Leftrightarrow Q) \Leftrightarrow R) \Leftrightarrow (P \Leftrightarrow (Q \Leftrightarrow R))$
distributivity	$R \wedge (P \vee Q) \Leftrightarrow (R \wedge P) \vee (R \wedge Q)$ $R \vee (P \wedge Q) \Leftrightarrow (R \vee P) \wedge (R \vee Q)$ $R \Rightarrow (P \wedge Q) \Leftrightarrow (R \Rightarrow P) \wedge (R \Rightarrow Q)$ $(P \vee Q) \Rightarrow R \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$

CLASSICAL RESULTS (2)

excluded middle	$P \vee \neg P$
idempotence	$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$
absorbtion	$(P \vee Q) \wedge P \Leftrightarrow P$ $(P \wedge Q) \vee P \Leftrightarrow P$
truth	$(P \Leftrightarrow \top) \Leftrightarrow P$
falsity	$(P \Leftrightarrow \bot) \Leftrightarrow \neg P$

CLASSICAL RESULTS (3)

de Morgan	$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ $\neg(P \wedge Q) \Leftrightarrow (P \Rightarrow \neg Q)$ $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
contraposition	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ $(\neg P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow P)$ $(P \Rightarrow \neg Q) \Leftrightarrow (Q \Rightarrow \neg P)$
double negation	$P \Leftrightarrow \neg\neg P$

CLASSICAL RESULTS (4)

transitivity	$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$
monotonicity	$(P \Rightarrow Q) \Rightarrow ((P \wedge R) \Rightarrow (Q \wedge R))$ $(P \Rightarrow Q) \Rightarrow ((P \vee R) \Rightarrow (Q \vee R))$ $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))$ $(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$ $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$
equivalence	$(P \Leftrightarrow Q) \Rightarrow ((P \wedge R) \Leftrightarrow (Q \wedge R))$ $(P \Leftrightarrow Q) \Rightarrow ((P \vee R) \Leftrightarrow (Q \vee R))$ $(P \Leftrightarrow Q) \Rightarrow ((R \Rightarrow P) \Leftrightarrow (R \Rightarrow Q))$ $(P \Leftrightarrow Q) \Rightarrow ((P \Rightarrow R) \Leftrightarrow (Q \Rightarrow R))$ $(P \Leftrightarrow Q) \Rightarrow (\neg P \Leftrightarrow \neg Q)$

Syntax of our Predicate Language so far

$$
\begin{array}{lll}
\mathit{predicate} & ::= & \bot \\
 & & \top \\
 & & \neg\, \mathit{predicate} \\
 & & \mathit{predicate} \wedge \mathit{predicate} \\
 & & \mathit{predicate} \vee \mathit{predicate} \\
 & & \mathit{predicate} \Rightarrow \mathit{predicate} \\
 & & \mathit{predicate} \Leftrightarrow \mathit{predicate}
\end{array}
$$

- The letter P, Q, etc. we have used are generic variables.
- Each of them stands for a *predicate*.
- All our proofs were thus also generic (able to be instantiated).

Refining our Language: Predicate Calculus

$$
\begin{array}{lll}
\mathit{predicate} & ::= & \bot \\
 & & \top \\
 & & \neg\, \mathit{predicate} \\
 & & \mathit{predicate} \wedge \mathit{predicate} \\
 & & \mathit{predicate} \vee \mathit{predicate} \\
 & & \mathit{predicate} \Rightarrow \mathit{predicate} \\
 & & \mathit{predicate} \Leftrightarrow \mathit{predicate} \\
 & & \forall \mathit{var\_list} \cdot \mathit{predicate} \\
 & & [\mathit{var\_list} := \mathit{exp\_list}]\, \mathit{predicate} \\
 & & \\
\mathit{expression} & ::= & \mathit{variable} \\
 & & [\mathit{var\_list} := \mathit{exp\_list}]\, \mathit{expression} \\
 & & \mathit{expression} \mapsto \mathit{expression} \\
 & & \\
\mathit{variable} & ::= & \mathit{identifier}
\end{array}
$$

On Predicates and Expressions

- A Predicate is a formal text that can be PROVED
- An Expression DENOTES AN OBJECT.
- A Predicate denotes NOTHING.
- An Expression CANNOT BE PROVED
- Predicates and Expressions are INCOMPATIBLE.

Predicate Calculus: Linguistic Concepts.

- Substitution and Universal Quantification.
- Free/Bound Occurrences.
- Inference rules.
- Extension

VARIABLES, PROPOSITIONS AND PREDICATES

- A Proposition: $8 \in \mathbb{N} \Rightarrow 8 \geq 0$
- A Predicate (n is a variable): $n \in \mathbb{N} \Rightarrow n \geq 0$

WHAT CAN WE DO WITH A PREDICATE ?

- Specialize it: Substitution

$$[n := 8]\,(n \in \mathbb{N} \Rightarrow n \geq 0)$$

$$\downarrow$$

$$8 \in \mathbb{N} \Rightarrow 8 \geq 0$$

- Generalize it: Universal Quantification

$$\forall n \cdot (n \in \mathbb{N} \Rightarrow n \geq 0)$$

SUBSTITUTION

Simple Substitution

$$[x := E]\,P$$

- x is a VARIABLE,
- E is an EXPRESSION,
- P is a PREDICATE,
- Denotes the predicate obtained by replacing all FREE OCCURRENCES of x by E in P.

UNIVERSAL QUANTIFICATION

Universal Quantification

$$\forall x \cdot P$$

- x is said to be the QUANTIFIED VARIABLE
- P forms the SCOPE of x
- To say that such a predicate is proved, is the same as saying that all predicates of the following form are proved:

$$[x := E]P$$

Free and Bound Occurrences

- Occurrences of the variable n are FREE (substitutable) in:

$$n \in \mathbb{N} \Rightarrow n \geq 0$$

- Occurrences of the variable n are BOUND (not substitutable) in:

$$[n := 8](n \in \mathbb{N} \Rightarrow n \geq 0)$$

$$\forall n \cdot (n \in \mathbb{N} \Rightarrow n \geq 0)$$

Inference Rules for Predicate Calculus

$$\frac{\mathsf{H},\ \forall x \cdot P,\ [x := E]P \ \vdash\ Q}{\mathsf{H},\ \forall x \cdot P \ \vdash\ Q}\ \textbf{ALL\_L}$$

where **E** is an expression

$$\frac{\mathbf{H} \ \vdash\ \mathbf{P}}{\mathbf{H} \ \vdash\ \forall \mathbf{x} \cdot \mathbf{P}}\ \textbf{ALL\_R}$$

- In rule ALL_R, variable **x** is not free in H

Extending the language: Existential Quantification

predicate	::=	$\bot$
		$\top$
		$\neg$ *predicate*
		predicate $\wedge$ *predicate*
		predicate $\vee$ *predicate*
		predicate $\Rightarrow$ *predicate*
		predicate $\Leftrightarrow$ *predicate*
		$\forall$ *var_list* $\cdot$ *predicate*
		$\exists$ *var_list* $\cdot$ *predicate*
		$[$*var_list* $:=$ *exp_list*$]$ *predicate*
expression	::=	*variable*
		$[$*var_list* $:=$ *exp_list*$]$ *expression*
		expression $\mapsto$ *expression*
variable	::=	*identifier*

Rules of Inference for Existential Quantification

$$\frac{\mathsf{H},\ P\ \vdash\ Q}{\mathsf{H},\ \exists x \cdot P\ \vdash\ Q}\quad \textbf{XST\_L}$$

- In rule XST_L, variable **x** is not free in **H** and **Q**

$$\frac{\mathsf{H}\ \vdash\ [x := E]P}{\mathsf{H}\ \vdash\ \exists x \cdot P}\quad \textbf{XST\_R}$$

where **E** is an expression

Comparing the Quantification Rules

$$\frac{\mathsf{H},\ \forall x \cdot P,\ [x := E]P\ \vdash\ Q}{\mathsf{H},\ \forall x \cdot P\ \vdash\ Q}\quad \textbf{ALL\_L}$$

$$\frac{\mathsf{H}\ \vdash\ [x := E]P}{\mathsf{H}\ \vdash\ \exists x \cdot P}\quad \textbf{XST\_R}$$

$$\frac{\mathsf{H}\ \vdash\ \mathsf{P}}{\mathsf{H}\ \vdash\ \forall x \cdot \mathsf{P}}\quad \textbf{ALL\_R}$$

$$\frac{\mathsf{H},\ P\ \vdash\ Q}{\mathsf{H},\ \exists x \cdot P\ \vdash\ Q}\quad \textbf{XST\_L}$$

CLASSICAL RESULTS (1)

commutativity	$\forall x \cdot \forall y \cdot P \ \Leftrightarrow\ \forall y \cdot \forall x \cdot P$ $\exists x \cdot \exists y \cdot P \ \Leftrightarrow\ \exists y \cdot \exists x \cdot P$
distributivity	$\forall x \cdot (P \wedge Q) \ \Leftrightarrow\ \forall x \cdot P \ \wedge\ \forall x \cdot Q$ $\exists x \cdot (P \vee Q) \ \Leftrightarrow\ \exists x \cdot P \ \vee\ \exists x \cdot Q$
associativity	if x not free in P $P \vee \forall x \cdot Q \ \Leftrightarrow\ \forall x \cdot (P \vee Q)$ $P \wedge \exists x \cdot Q \ \Leftrightarrow\ \exists x \cdot (P \wedge Q)$ $P \Rightarrow \forall x \cdot Q \ \Leftrightarrow\ \forall x \cdot (P \Rightarrow Q)$

CLASSICAL RESULTS (2)

de Morgan laws	$\neg \forall x \cdot P \ \Leftrightarrow\ \exists x \cdot \neg P$ $\neg \exists x \cdot P \ \Leftrightarrow\ \forall x \cdot \neg P$ $\neg \forall x \cdot (P \Rightarrow Q) \ \Leftrightarrow\ \exists x \cdot (P \wedge \neg Q)$ $\neg \exists x \cdot (P \wedge Q) \ \Leftrightarrow\ \forall x \cdot (P \Rightarrow \neg Q)$
monotonicity	$\forall x \cdot (P \Rightarrow Q) \ \Rightarrow\ (\forall x \cdot P \ \Rightarrow\ \forall x \cdot Q)$ $\forall x \cdot (P \Rightarrow Q) \ \Rightarrow\ (\exists x \cdot P \ \Rightarrow\ \exists x \cdot Q)$
equivalence	$\forall x \cdot (P \Leftrightarrow Q) \ \Rightarrow\ (\forall x \cdot P \ \Leftrightarrow\ \forall x \cdot Q)$ $\forall x \cdot (P \Leftrightarrow Q) \ \Rightarrow\ (\exists x \cdot P \ \Leftrightarrow\ \exists x \cdot Q)$

Summary of Logical Operators

$P \wedge Q$	$\neg P$
$P \vee Q$	$\forall x \cdot P$
$P \Rightarrow Q$	$\exists x \cdot P$

Refining our Language: Equality

$$
\begin{array}{lll}
\textit{predicate} & ::= & \bot \\
 & & \top \\
 & & \neg\ \textit{predicate} \\
 & & \textit{predicate} \wedge \textit{predicate} \\
 & & \textit{predicate} \vee \textit{predicate} \\
 & & \textit{predicate} \Rightarrow \textit{predicate} \\
 & & \textit{predicate} \Leftrightarrow \textit{predicate} \\
 & & \forall \textit{variable} \cdot \textit{predicate} \\
 & & \exists \textit{variable} \cdot \textit{predicate} \\
 & & [\textit{variable} := \textit{expression}]\ \textit{predicate} \\
 & & \textit{expression} = \textit{expression} \\
\textit{expression} & ::= & \cdots \\
\textit{variable} & ::= & \cdots
\end{array}
$$

Equality Rules of Inference

$$\frac{[x := F]\mathsf{H},\ E = F \ \vdash\ [x := F]P}{[x := E]\mathsf{H},\ E = F \ \vdash\ [x := E]P}\ \textbf{EQ\_LR}$$

$$\frac{[x := E]\mathsf{H},\ E = F \ \vdash\ [x := E]P}{[x := F]\mathsf{H},\ E = F \ \vdash\ [x := F]P}\ \textbf{EQ\_RL}$$

- Rewriting rules:

Operator	Predicate	Rewritten
Equality	$E = E$	$\top$
Equality of pairs	$E \mapsto F \ = \ G \mapsto H$	$E = G \ \wedge \ F = H$

Classical Results for Equality

symmetry	$E = F \ \Leftrightarrow \ F = E$
transitivity	$E = F \ \wedge \ F = G \ \Rightarrow \ E = G$
One-point rules	if $\boldsymbol{x}$ not free in $\boldsymbol{E}$
	$\forall x \cdot (x = E \ \Rightarrow \ P) \ \Leftrightarrow \ [x := E]P$
	$\exists x \cdot (x = E \ \wedge \ P) \ \Leftrightarrow \ [x := E]P$

Refining our Language: Set Theory (1)

$$
\begin{array}{lll}
\textit{predicate} & ::= & \bot \\
& & \top \\
& & \neg\, \textit{predicate} \\
& & \textit{predicate} \wedge \textit{predicate} \\
& & \textit{predicate} \vee \textit{predicate} \\
& & \textit{predicate} \Rightarrow \textit{predicate} \\
& & \textit{predicate} \Leftrightarrow \textit{predicate} \\
& & \forall\, \textit{var\_list} \cdot \textit{predicate} \\
& & \exists\, \textit{var\_list} \cdot \textit{predicate} \\
& & [\textit{var\_list} := \textit{exp\_list}]\, \textit{predicate} \\
& & \textit{expression} = \textit{expression} \\
& & \textit{expression} \in \textit{set}
\end{array}
$$

Refining our Language: Set Theory (2)

$$
\begin{array}{lll}
\textit{expression} & ::= & \textit{variable} \\
& & [\textit{var\_list} := \textit{exp\_list}]\, \textit{expression} \\
& & \textit{expression} \mapsto \textit{expression} \\
& & \textit{set} \\
\textit{variable} & ::= & \textit{identifier} \\
\textit{set} & ::= & \textit{set} \times \textit{set} \\
& & \mathbb{P}(\textit{set}) \\
& & \{\, \textit{var\_list} \cdot \textit{predicate} \mid \textit{expression} \,\}
\end{array}
$$

- When *expression* is the same as *var_list*, the last construct can be written $\{\, \textit{var\_list} \mid \textit{predicate} \,\}$

Set Theory

1. Basis
   - Basic operators
2. Extensions
   - Elementary operators
   - Generalization of elementary operators
   - Binary relation operators
   - Function operators

Set Theory: Membership

- Set theory deals with a new predicate: the membership predicate

$$E \in S$$

where E is an expression and S is a set

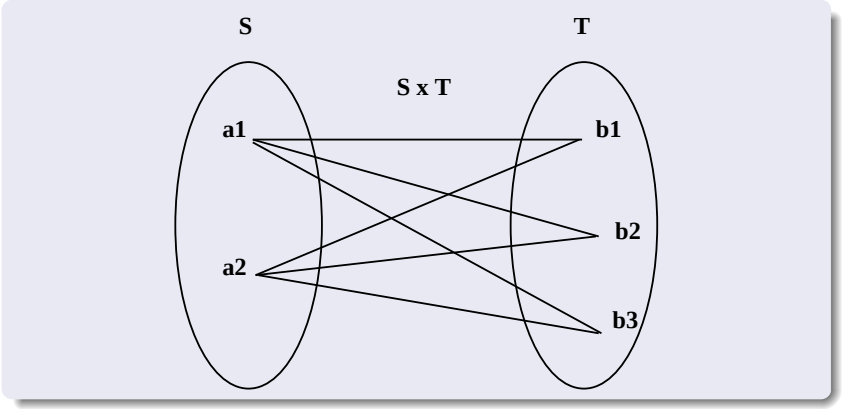
Set Theory: Basic Constructs

There are three basic constructs in set theory:

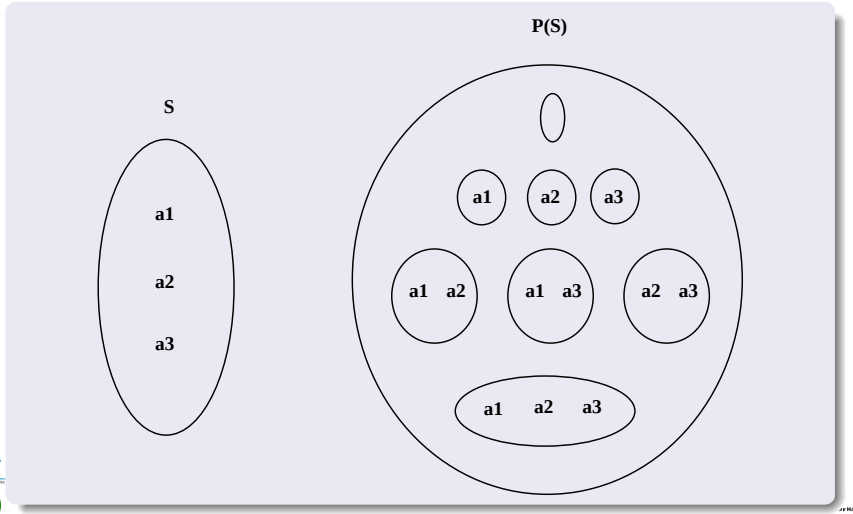
Cartesian product	$S \times T$
Power set	$\mathbb{P}(S)$
Comprehension 1	$\{x \cdot P \mid F\}$
Comprehension 2	$\{x \mid P\}$

where S and T are sets, x is a variable and P is a predicate.

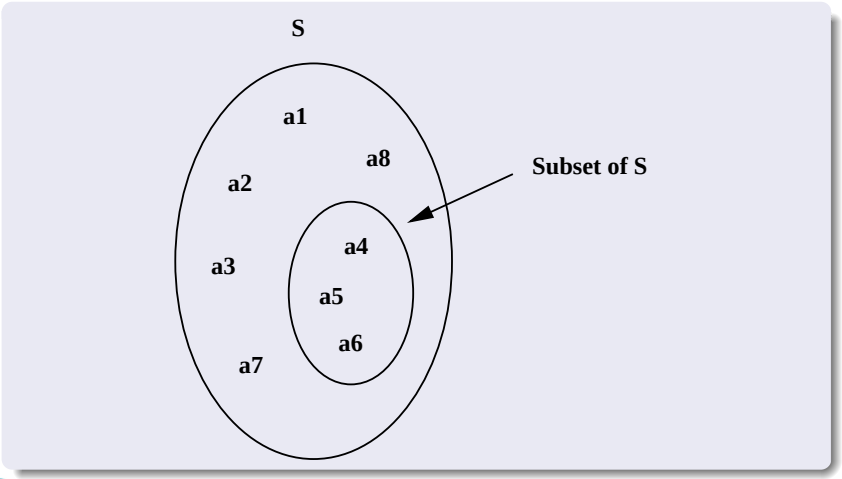
Cartesian Product



Power Set



Set Comprehension



Basic Set Operator Memberships (Axioms)

These axioms are defined by equivalences.

Left Part	Right Part
$E \mapsto F \in S \times T$	$E \in S \ \wedge \ F \in T$
$S \in \mathbb{P}(T)$	$\forall x \cdot (x \in S \Rightarrow x \in T)$ (x is not free in S and T)
$E \in \{x \cdot P \mid F\}$	$\exists x \cdot P \ \wedge \ E = F$ (x is not free in E)
$E \in \{x \mid P\}$	$[x := E]P$ (x is not free in E)

Set Inclusion and Extensionality Axiom

Left Part	Right Part
$S \subseteq T$	$S \in \mathbb{P}(T)$
$S = T$	$S \subseteq T \ \wedge \ T \subseteq S$

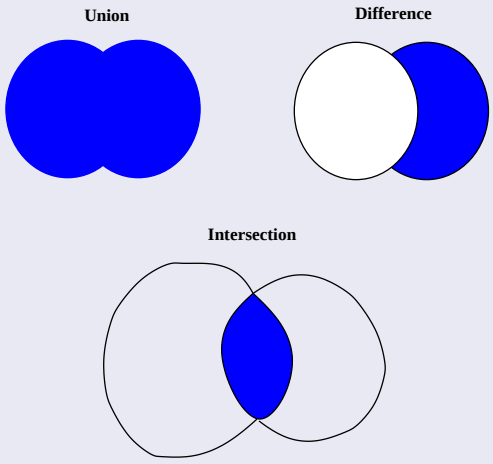
The first rule is just a syntactic extension

The second rule is the Extensionality Axiom

Elementary Set Operators

Union	$S \cup T$
Intersection	$S \cap T$
Difference	$S \setminus T$
Extension	$\{a, \ldots, b\}$
Empty set	$\varnothing$

Union, Difference, Intersection



Elementary Set Operator Memberships

$E \in S \cup T$	$E \in S \ \vee \ E \in T$
$E \in S \cap T$	$E \in S \ \wedge \ E \in T$
$E \in S \setminus T$	$E \in S \ \wedge \ E \notin T$
$E \in \{a, \ldots, b\}$	$E = a \ \vee \ \ldots \ \vee \ E = b$
$E \in \varnothing$	$\bot$

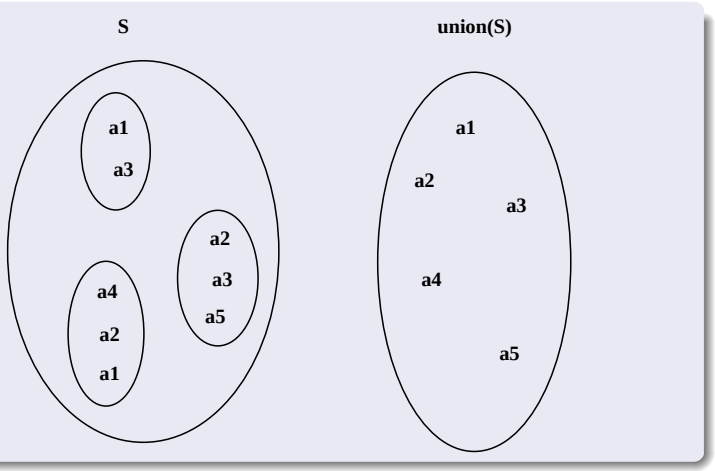
Summary of Basic and Elementary Operators

$S \times T$	$S \cup T$
$\mathbb{P}(S)$	$S \cap T$
$\{x \cdot P \mid F\}$	$S \setminus T$
$S \subseteq T$	$\{a, \ldots, b\}$
$S = T$	$\varnothing$

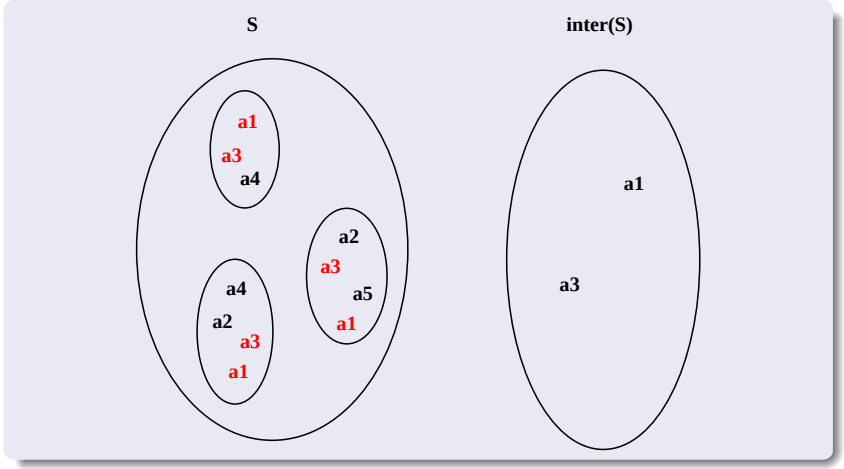
Generalizations of Elementary Operators

Generalized Union	$\mathrm{union}\,(S)$
Union Quantifier	$\bigcup x \cdot (P \mid T)$
Generalized Intersection	$\mathrm{inter}\,(S)$
Intersection Quantifier	$\bigcap x \cdot (P \mid T)$

Generalized Union



Generalized Intersection



Generalizations of Elementary Operator Memberships

$E \in \text{union}(S)$	$\exists s \cdot s \in S \wedge E \in s$ (s is not free in S and E)
$E \in (\bigcup x \cdot P \mid T)$	$\exists x \cdot P \wedge E \in T$ (x is not free in E)
$E \in \text{inter}(S)$	$\forall s \cdot s \in S \Rightarrow E \in s$ (s is not free in S and E)
$E \in (\bigcap x \cdot P \mid T)$	$\forall x \cdot P \Rightarrow E \in T$ (x is not free in E)

Well-definedness condition for case 3: $S \neq \varnothing$
Well-definedness condition for case 4: $\exists x \cdot P$

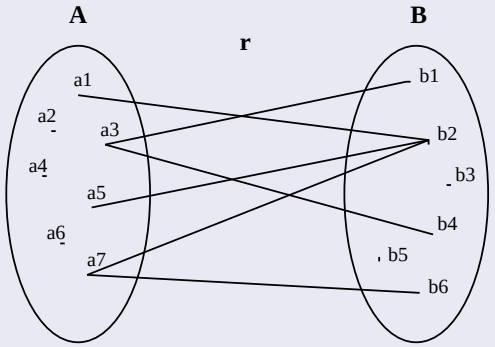
Summary of Generalizations of Elementary Operators

$\text{union}(S)$
$\bigcup x \cdot P \mid T$
$\text{inter}(S)$
$\bigcap x \cdot P \mid T$

Binary Relation Operators (1)

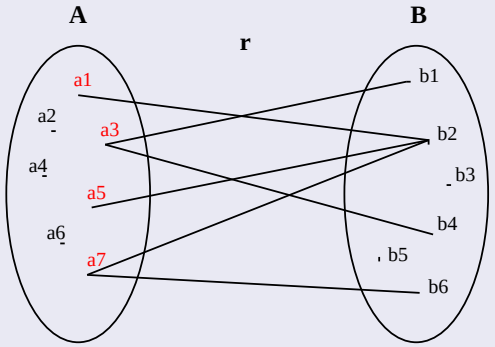
Binary relations	$S \leftrightarrow T$
Domain	$\text{dom}(r)$
Range	$\text{ran}(r)$
Converse	r^{-1}

A Binary Relation r from a Set A to a Set B



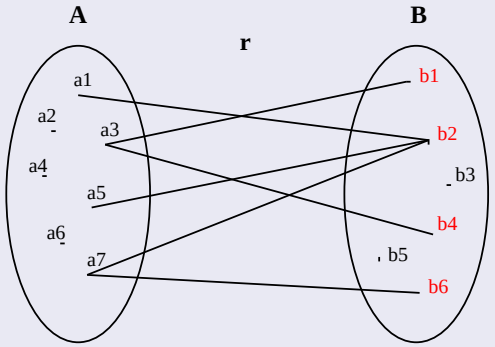
$r \in A \leftrightarrow B$

Domain of Binary Relation r



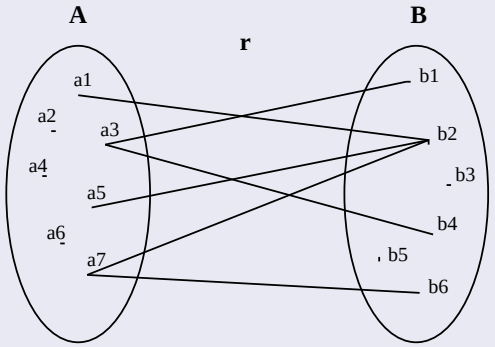
$\mathrm{dom}(r) = \{a1, a3, a5, a7\}$

Range of Binary Relation r



$\mathrm{ran}(r) = \{b1, b2, b4, b6\}$

Converse of Binary Relation r



$r^{-1} = \{b1 \mapsto a3, b2 \mapsto a1, b2 \mapsto a5, b2 \mapsto a7, b4 \mapsto a3, b6 \mapsto a7\}$

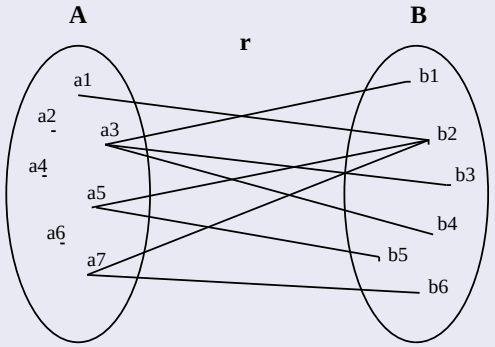
Binary Relation Operator Memberships (1)

Left Part	Right Part
$r \in S \leftrightarrow T$	$r \subseteq S \times T$
$E \in \mathrm{dom}\,(r)$	$\exists y \cdot E \mapsto y \in r$ (y is not free in E and r)
$F \in \mathrm{ran}\,(r)$	$\exists x \cdot x \mapsto F \in r$ (x is not free in F and r)
$E \mapsto F \in r^{-1}$	$F \mapsto E \in r$

Binary Relation Operators (2)

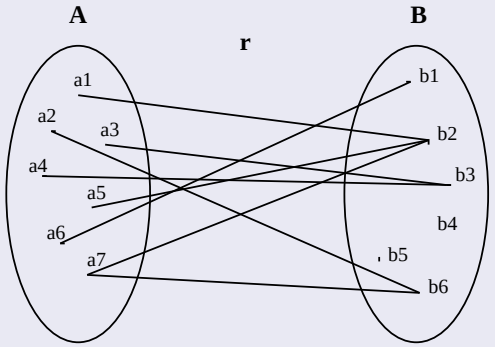
Partial surjective binary relations	$S \leftrightarrow\!\!\!\!\rightarrow T$
Total binary relations	$S \leftrightarrow\!\!\!\!\!\rightarrow T$
Total surjective binary relations	$S \leftrightarrow\!\!\!\!\!\!\rightarrow T$

A Partial Surjective Relation



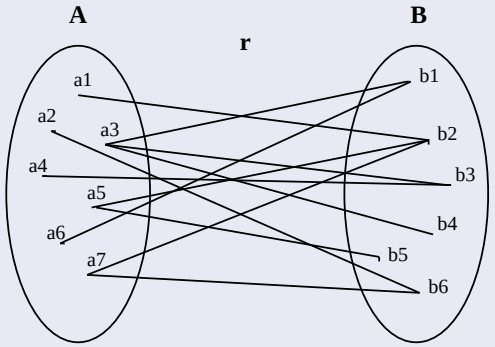
$r \in A \leftrightarrow\!\!\!\!\rightarrow B$

A Total Relation



$r \in A \leftrightarrow\!\!\!\!\!\rightarrow B$

A Total Surjective Relation



$r \in A \twoheadrightarrow B$

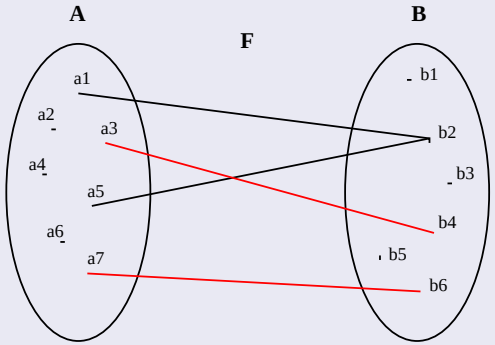
Binary Relation Operator Memberships (2)

Left Part	Right Part
$r \in S \leftrightarrow\!\!\!\!\!\rightarrow T$	$r \in S \leftrightarrow T \ \wedge \ \mathrm{ran}(r) = T$
$r \in S \leftarrow\!\!\!\!\!\leftrightarrow T$	$r \in S \leftrightarrow T \ \wedge \ \mathrm{dom}(r) = S$
$r \in S \leftarrow\!\!\!\!\!\rightarrow T$	$r \in S \leftrightarrow T \ \wedge \ r \in S \leftrightarrow\!\!\!\!\!\rightarrow T$

Binary Relation Operators (3)

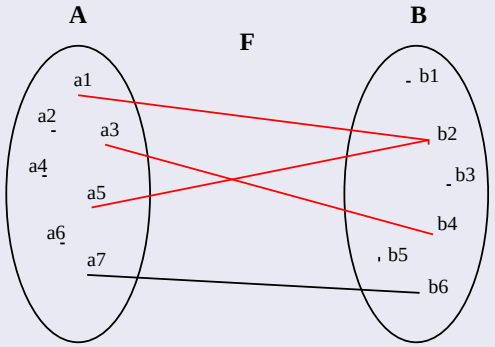
Domain restriction	$S \lhd r$
Range restriction	$r \rhd T$
Domain subtraction	$S \lhd\!\!\!- r$
Range subtraction	$r \rhd\!\!\!- T$

The Domain Restriction Operator



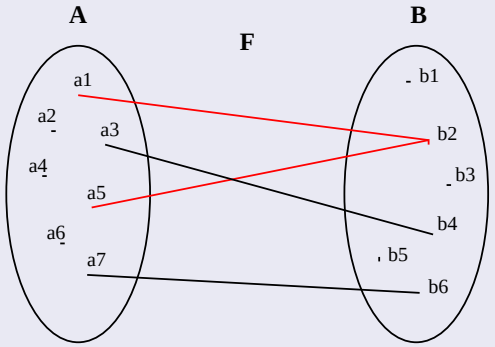
$\{a3,\ a7\} \lhd F$

The Range Restriction Operator



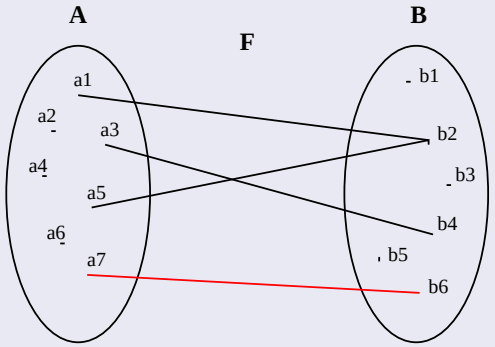
$F \rhd \{b2, b4\}$

The Domain Substraction Operator



$\{a3,\ a7\} \lhd F$

The Range Substraction Operator



$F \rhd \{b2, b4\}$

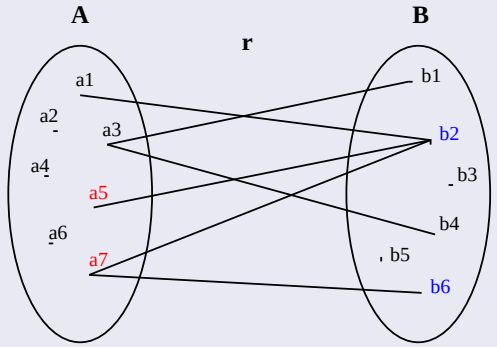
Binary Relation Operator Memberships (3)

Left Part	Right Part
$E \mapsto F \in S \lhd r$	$E \in S \ \wedge \ E \mapsto F \in r$
$E \mapsto F \in r \rhd T$	$E \mapsto F \in r \ \wedge \ F \in T$
$E \mapsto F \in S \lhd\!\!\!\!- r$	$E \notin S \ \wedge \ E \mapsto F \in r$
$E \mapsto F \in r \rhd\!\!\!\!- T$	$E \mapsto F \in r \ \wedge \ F \notin T$

Binary Relation Operators (4)

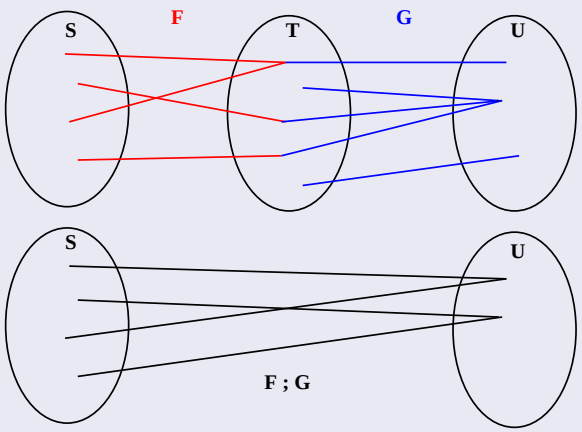
Image	$r[w]$
Composition	$p \,;\, q$
Overriding	$p \lhd q$
Identity	$\mathrm{id}\,(S)$

Image of $\{a5, a7\}$ under r

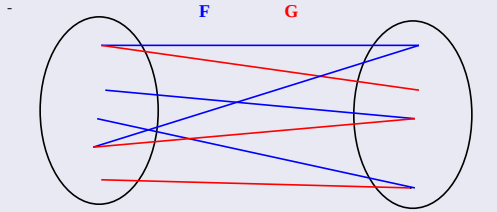


$r[\{a5, a7\}] = \{b2, b6\}$

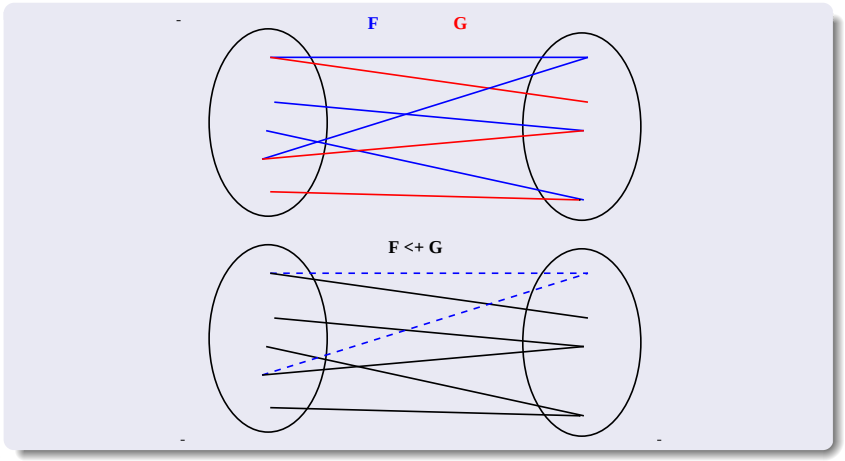
Forward Composition



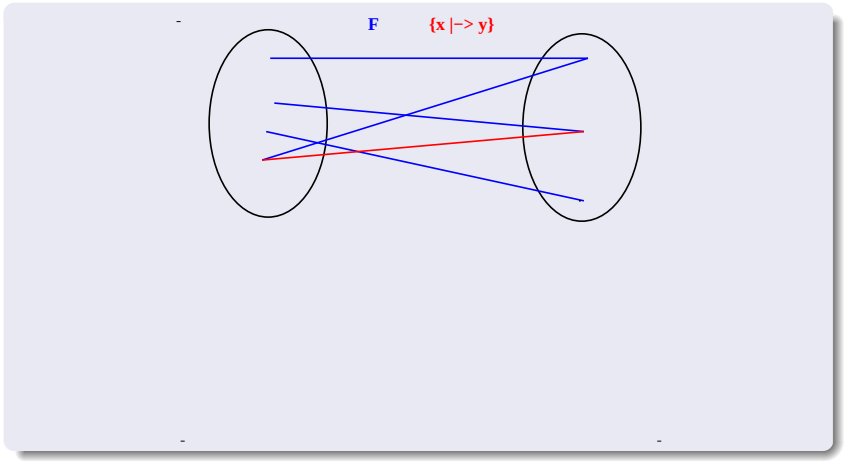
The Overriding Operator



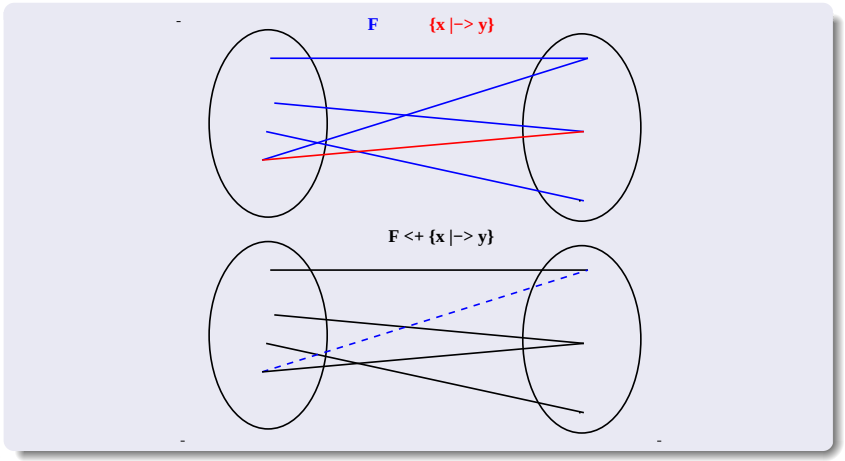
The Overriding Operator



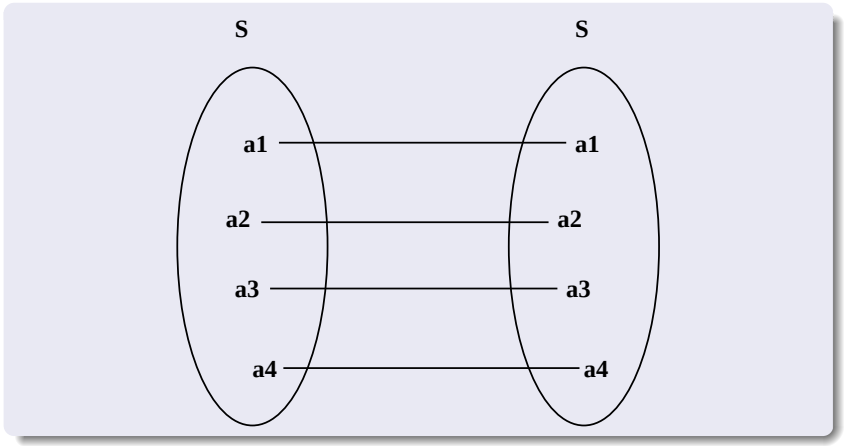
Special Case



Special Case



The Identity Relation



Binary Relation Operator Memberships (4)

$F \in r[w]$	$\exists x \cdot x \in w \ \wedge \ x \mapsto F \in r$ (x is not free in F, r and w)
$E \mapsto F \in (p\,;q)$	$\exists x \cdot E \mapsto x \in p \ \wedge \ x \mapsto F \in q$ (x is not free in E, F, p and q)
$E \mapsto F \in p \lhd\!\!\!- q$	$E \mapsto F \in (\mathrm{dom}\,(q) \lhd p) \ \cup \ q$
$E \mapsto F \in \mathrm{id}\,(S)$	$E \in S \ \wedge \ F = E$

Binary Relation Operators (5)

Direct Product	$p \otimes q$
First Projection	$\mathrm{prj}_1(S, T)$
Second Projection	$\mathrm{prj}_2(S, T)$
Parallel Product	$p \parallel q$

Binary Relation Operator Memberships (5)

$E \mapsto (F \mapsto G) \in p \otimes q$	$E \mapsto F \in p \ \wedge \ E \mapsto G \in q$
$(E \mapsto F) \mapsto G \in \mathrm{prj}_1(S, T)$	$E \in S \ \wedge \ F \in T \ \wedge \ G = E$
$(E \mapsto F) \mapsto G \in \mathrm{prj}_2(S, T)$	$E \in S \ \wedge \ F \in T \ \wedge \ G = F$
$(E \mapsto G) \mapsto (F \mapsto H) \in p \parallel q$	$E \mapsto F \in p \ \wedge \ G \mapsto H \in q$

Summary of Binary Relation Operators

$S \leftrightarrow T$	$S \lhd r$	$r[w]$	$\mathrm{prj}_1\,(S, T)$
$\mathrm{dom}\,(r)$	$r \rhd T$	$p\,;q$	$\mathrm{prj}_2\,(S, T)$
$\mathrm{ran}\,(r)$	$S \lhd\!\!\!- r$	$p \lhd\!\!\!- q$	$\mathrm{id}\,(S)$
r^{-1}	$r \rhd\!\!\!- T$	$p \otimes q$	$p \parallel q$

Classical Results with Relation Operators

$$r^{-1-1} = r$$

$$\mathrm{dom}(r^{-1}) = \mathrm{ran}(r)$$

$$(S \lhd r)^{-1} = r^{-1} \rhd S$$

$$(p\,;q)^{-1} = q^{-1}\,;p^{-1}$$

$$(p\,;q)\,;r = q\,;(p\,;r)$$

$$(p\,;q)[w] = q[p[w]]$$

$$p\,;(q \cup r) = (p\,;q) \cup (p\,;r)$$

$$r[a \cup b] = r[a] \cup r[b]$$

More classical Results

Given a relation r such that $r \in S \leftrightarrow S$

$r = r^{-1}$	r is symmetric
$r \cap r^{-1} = \varnothing$	r is asymmetric
$r \cap r^{-1} \subseteq \mathrm{id}(S)$	r is antisymmetric
$\mathrm{id}(S) \subseteq r$	r is reflexive
$r \cap \mathrm{id}(S) = \varnothing$	r is irreflexive
$r;r \subseteq r$	r is transitive

Translations into First Order Predicates

Given a relation r such that $r \in S \leftrightarrow S$

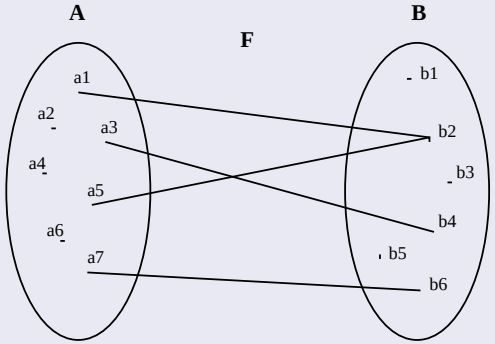
$r = r^{-1}$	$\forall x, y \cdot x \in S \land y \in S \Rightarrow (x \mapsto y \in r \Leftrightarrow y \mapsto x \in r)$
$r \cap r^{-1} = \varnothing$	$\forall x, y \cdot x \mapsto y \in r \Rightarrow y \mapsto x \notin r$
$r \cap r^{-1} \subseteq \mathrm{id}(S)$	$\forall x, y \cdot x \mapsto y \in r \land y \mapsto x \in r \Rightarrow x = y$
$\mathrm{id}(S) \subseteq r$	$\forall x \cdot x \in S \Rightarrow x \mapsto x \in r$
$r \cap \mathrm{id}(S) = \varnothing$	$\forall x, y \cdot x \mapsto y \in r \Rightarrow x \neq y$
$r;r \subseteq r$	$\forall x, y, z \cdot x \mapsto y \in r \land y \mapsto z \in r \Rightarrow x \mapsto z \in r$

Set-theoretic statements are far more readable than predicate calculus statements

Function Operators (1)

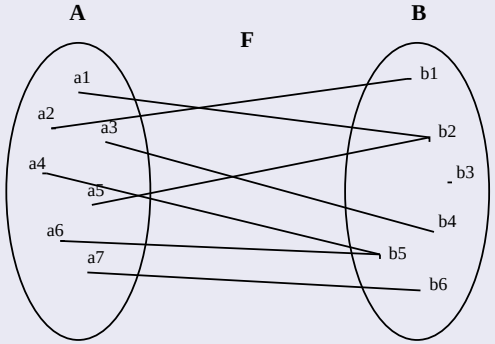
Partial functions	$S \nrightarrow T$
Total functions	$S \rightarrow T$
Partial injections	$S \rightarrowtail T$
Total injections	$S \rightarrowtail T$

A Partial Function F from a Set A to a Set B



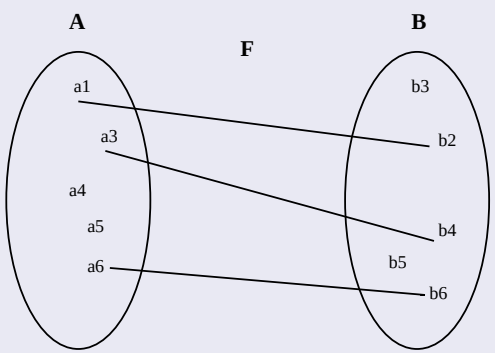
$F \in A \nrightarrow B$

A Total Function F from a Set A to a Set B



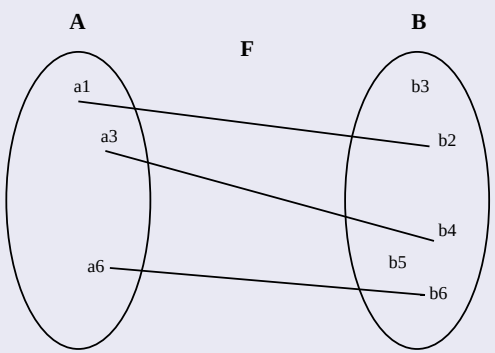
$F \in A \rightarrow B$

A Partial Injection F from a Set A to a Set B



$F \in A \rightarrowtail B$

A Total Injection F from a Set A to a Set B



$F \in A \rightarrowtail B$

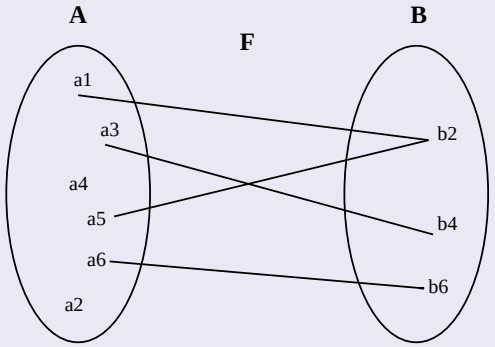
Function Operator Memberships (1)

Left Part	Right Part
$f \in S \nrightarrow T$	$f \in S \leftrightarrow T \ \wedge \ (f^{-1}; f) = \mathrm{id}(\mathrm{ran}(f))$
$f \in S \rightarrow T$	$f \in S \nrightarrow T \ \wedge \ s = \mathrm{dom}(f)$
$f \in S \rightarrowtail T$	$f \in S \nrightarrow T \ \wedge \ f^{-1} \in T \nrightarrow S$
$f \in S \rightarrowtail T$	$f \in S \rightarrow T \ \wedge \ f^{-1} \in T \nrightarrow S$

Function Operators (2)

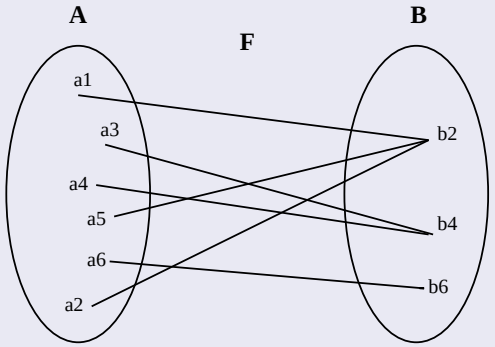
Partial surjections	$S \nrightarrow\!\!\!\!\rightarrow T$
Total surjections	$S \rightarrow\!\!\!\!\rightarrow T$
Bijections	$S \rightarrowtail\!\!\!\!\rightarrow T$

A Partial Surjection F from a Set A to a Set B



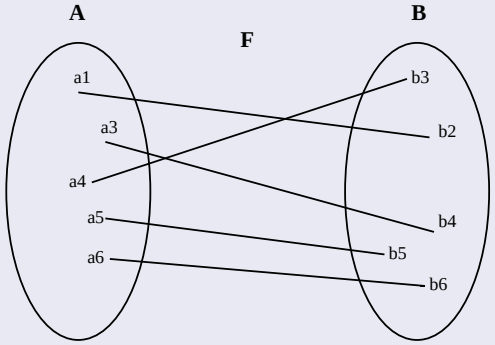
$F \in A \nrightarrow\!\!\!\!\rightarrow B$

A Total Surjection F from a Set A to a Set B



$F \in A \rightarrow\!\!\!\!\rightarrow B$

A Bijection F from a Set A to a Set B



$F \in A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$

Function Operator Memberships (2)

Left Part	Right Part
$f \in S \twoheadrightarrow T$	$f \in S \mathrel{+\!\!\!\rightarrow} T \ \wedge \ T = \mathrm{ran}(f)$
$f \in S \rightarrow\!\!\!\!\!\twoheadrightarrow T$	$f \in S \rightarrow T \ \wedge \ T = \mathrm{ran}(f)$
$f \in S \rightarrowtail\!\!\!\!\!\twoheadrightarrow T$	$f \in S \rightarrowtail T \ \wedge \ f \in S \rightarrow\!\!\!\!\!\twoheadrightarrow T$

Summary of Function Operators

$S \mathrel{+\!\!\!\rightarrow} T$	$S \twoheadrightarrow T$
$S \rightarrow T$	$S \rightarrow\!\!\!\!\!\twoheadrightarrow T$
$S \rightarrowtail T$	$S \rightarrowtail\!\!\!\!\!\twoheadrightarrow T$
$S \rightarrowtail T$	

Summary of all Set-theoretic Operators (40)

$S \times T$	$S \setminus T$	r^{-1}	$r[w]$	$\mathrm{id}(S)$	$\{x \mid x \in S \ \wedge \ P\}$
$\mathbb{P}(S)$	$S \leftrightarrow T$ $S \mathrel{+\!\!\!\leftrightarrow} T$	$S \lhd r$ $S \mathrel{\lhd\!\!\!-} r$	$p \,;\, q$	$S \mathrel{+\!\!\!\rightarrow} T$ $S \rightarrow T$	$\{x \cdot x \in S \ \wedge \ P \mid E\}$
$S \subseteq T$	$S \leftrightarrow\!\!\!\!\!\twoheadrightarrow T$ $S \mathrel{+\!\!\!\leftrightarrow} T$	$r \rhd T$ $r \mathrel{\rhd\!\!\!-} T$	$p \mathrel{\triangleleft\!\!\!-} q$	$S \rightarrowtail\!\!\!\!\!\twoheadrightarrow T$ $S \rightarrowtail T$	$\{a, b, \ldots, n\}$
$S \cup T$	$\mathrm{dom}(r)$ $\mathrm{ran}(r)$	prj_1	$p \otimes q$	$S \twoheadrightarrow T$ $S \rightarrow T$	union $\bigcup$
$S \cap T$	$\varnothing$	prj_2	$p \parallel q$	$S \rightarrowtail\!\!\!\!\!\twoheadrightarrow T$	inter $\bigcap$

Applying a Function

Given a partial function f, we have

Left Part	Right Part
$F = f(E)$	$E \mapsto F \in f$

Well-definedness condition: $E \in \mathrm{dom}(f)$

Example: a Very Strict Society

- Every person is either a man or a woman
- But no person can be a man and a woman at the same time
- Only women have husbands, who must be a man
- Woman have at most one husband
- Likewise, men have at most one wife
- Moreover, mother are married women

Formal Representation

$$men \subseteq PERSON$$

$$women = PERSON \setminus men$$

$$husband \in women \rightarrowtail\!\!\!\!\twoheadrightarrow men$$

$$mother \in PERSON \rightarrow \mathrm{dom}(husband)$$

- Every person is either a man or a woman.
- But no person can be a man and a woman at the same time.
- Only women have husbands, who must be a man.
- Woman have at most one husband.
- Likewise, men have at most one wife.
- Moreover, mother are married women.

Defining New Concepts

$$men \subseteq PERSON$$

$$women = PERSON \setminus men$$

$$husband \in women \rightarrowtail\!\!\!\!\twoheadrightarrow men$$

$$mother \in PERSON \rightarrow \mathrm{dom}(husband)$$

$$wife = husband^{-1}$$

$$spouse = husband \cup wife$$

$$father = mother \; ; husband$$

Defining New Concepts

$$men \subseteq PERSON$$

$$women = PERSON \setminus men$$

$$husband \in women \rightarrowtail men$$

$$mother \in PERSON \rightarrow \mathrm{dom}(husband)$$

$$father = mother ; husband$$

$$children = (mother \cup father)^{-1}$$

$$daughter = children \rhd women$$

$$sibling = (children^{-1} ; children) \setminus \mathrm{id}(PERSON)$$

Exercises. To be defined

$$brother = ?$$

$$sibling - in - law = ?$$

$$nephew - or - niece = ?$$

$$uncle - or - aunt = ?$$

$$cousin = ?$$

Exercises. To be proved

$$mother = father ; wife$$

$$spouse = spouse^{-1}$$

$$sibling = sibling^{-1}$$

$$cousin = cousin^{-1}$$

$$father ; father^{-1} = mother ; mother^{-1}$$

$$father ; mother^{-1} = \varnothing$$

$$mother ; father^{-1} = \varnothing$$

$$father ; children = mother ; children$$

For Further Reading I

J-R. Abrial.
Modeling in Event-B: System and Software Engineering, Chapter 9 — Mathematical Language.
CUP, 2010.