The frame alignment problem in formations of multi-agent systems ⋆

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Abstract: Problems dealing with coordination of autonomous vehicles usually assume that the agents share a universal orientation in their local reference frames. Although the sensing requirements to satisfy this assumption can sometimes be low, they cannot always be guaranteed. The basic problem of frame alingment is studied here, in the absence of orientation sensors, by exploiting the notions of rigidity, visibility and communication graphs. First the relationship is examined between geometric rigidity of a formation and the agents' ability to agree on a shared orientation to their local frames. Our main results are then formulated in terms of visibility and communication graphs using cyclic graph coverage. Cases when only distances are measurable are also investigated, in which case the motion capabilities of the agents need to be exploited to resolve ambiguity caused by symmetry.

Keywords: Autonomous control, autonomous vehicles, distributed control, global coordinate system, frame alignment problem, communication graphs, visibility graphs.

1. INTRODUCTION

Developments in distributed sensing have received widespread interest in recent years due, the appearance of inexpensive sensors and the correspondent developments in distributed control systems. These systems promise superior performance and lower costs relative to their expensive high performance single units of sensing with the added benefits of robustness. Mounting these small sensing platforms into movable platforms provides a network which, coupled with some intelligence, can react to the conditions of the environment.

Ideally we would like each of these mobile platforms to take decisions by taking into account what is happening around them (local decisions) and to react in a way that improves the overall global performance of the system. Even though the field of autonomous control for mobile systems has been active during the last decade, fundamental problems remain open. One of those fundamental problems is frame alignment: it is usually assumed that all the agents understand their relative motions and positions with respect to each other. If there is no attitude sensor available then seeing each other and other non-agent environmental objects can help this alignment of frames. If their visibility graph (see definitions later) contains rigid structures, they can use those for frame alignment. Hence rigidity of formations is relevant to the frame alignment problem. For 2D deployments, rigidity theory (see for instance Krick (2007) and references therein) provides a satisfactory solution. However, in higher dimensions, there is still no set of sufficient and necessary conditions that guarantee a particular arrangement is rigid based on the number of pair wise distances specified. Laman's condition, as generalized to 3D, has proven insufficient (Mantler and Snoeyink (2004)) for rigidity. More recent publications Kim et al. (2010); Giordano et al. (2011) have not solved this problem yet but addressed the mechanics and control of rigid formations.

We have recently explored an alternative approach to frame alignment in a recent paper (Caicedo-N and Veres (2013)) when the particular task that the agents had to address was formation control (Fax and Murray (2004),Jadbabaie et al. (2003),Moreau (2004),Olfati-Saber and Murray (2004)). We exploited consensus protocols to reach agreement on local frame orientation of the reference frame when the underlying communication graph was connected. Our analysis has resulted that using a consensus control law no agreement on the orientation would lead the agents towards a stationary configuration. In this paper we examine how rigidity can help to address this issue. We present a condition that ensures rigidity (and hence leads to frame alignment under sufficient communications links between agents), and discuss restrictions on the possible length of a cycle that can be induced by the communication graph induced by the robots.

The paper is organized as follows: §2 presents an overview on the terminology and notation used throughout, as well as a brief overview on rigidity; §3 discusses how can the agents agree on a reference frame orientation by relying on distances alone, and compares it to earlier results in the field (in particular, that in Asimow and Roth (1978)); §4

⋆ This research was supported by EPSRC grant EP/J011843/1.

discusses how the agents capabilities can be exploited to extend the condition from §3; and §5 concludes.

2. PRELIMINARIES AND NOTATION

Consider N agents, labelled $\{1, 2, \ldots, N\}$ deployed in $\mathbb{R}^3$. Let $\mathcal{G}_C = (\mathcal{V}, \mathcal{E})$ denote a *communication graph* with vertices in 1-1 correspondence with the agents and $(i, j) \in \mathcal{E}$ if and only if agents i and j can communicate with each other. We assume that communication is a symmetric operation and, henceforth, $(i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}$. Additionally, consider a directed bipartite *visibility graph* $\mathcal{G}_V = (\mathcal{V}, \mathcal{O}, \mathcal{S})$ where $\mathcal{V}$ represents the agents and $\mathcal{O}$ is for a set of environmental objects that are not agents. If there is a directed edge from an agent vertex in $\mathcal{V}$ to an object vertex in $\mathcal{O}$ then the agent can see the object. Similarly, when agents can see each other, that is represented by directed edges in $\mathcal{S}$. In this paper we assume that such graphs satisfy the following conditions.

Assumption 1. (1) The visibility relation $\mathcal{S}$ is symmetric over the agent set $\mathcal{V}$ in $\mathcal{G}_V$.
(2) The visibility relation $\mathcal{S}$ has directed edges from $\mathcal{V}$ to $\mathcal{O}$ and none over $\mathcal{O}$, or from $\mathcal{O}$ into $\mathcal{V}$ in $\mathcal{G}_V$.
(3) The communication graph in $\mathcal{G}_C$ is symmetric and connected.

The first condition implies mutual visibility between agents based on similar sensing capabilities. The second assumption states that only an agent possesses the ability of vision that means bearing measurements in local frames. The third assumption is to support each agent's ability to store transmitted information about other agents indirectly. We thus assume, without loss of generality, that each agent has a complete list of all other agents in the formation, each agent can know when an agent enters/leaves the formation (as in Caicedo-N and Žefran (2010) that allows them to know information about other agents they do not communicate with), so the information gathered by agent i and shared to agent j, might be shared from agent j to other agents that are not neighbours of i. Connectivity enables the distribution of all information to all agents, using each agent to re-broadcast any information it receives to all those it communicates with. The notion of connectivity in $\mathcal{G}_C$ leads to the definition of neighbours.

Definition 1. For an agent with index i its neighbouring set $\mathcal{N}_i$ is defined by

$$\mathcal{N}_i = \{j |\ (i, j) \in \mathcal{E}\}$$

and we call the elements of this set, the *neighbours of* i.

The concept of visibility needs to be defined to give a more precise meaning to the presence of an edge in $\mathcal{G}_V$.

Definition 2. A directed edge $e \in \mathcal{S}$ connecting agents i and j means that i has a body frame coordinate system in which it can determine the position of j. In this case, we say that agent i can *see* agent j. We also say that agent i *vaguely sees* agent j if it can determine its distance but not its precise direction in its local coordinate system.

Inclusion of vague visibility extends $\mathcal{G}_V$ into a labelled graph. The visibility relation is however not assumed to be transitive. The practical reason for this is that beyond a certain distance the viewing agent is not be able to identify the position of the agent seen in its coordinate for that to be useful in shaping a geometric formation.

Finally we define a sufficient condition between geometric rigidity of a formation of agents and their ability to define local coordinates frames with aligned orientations.

Lemma 1. [*sufficient condition of align ability*] If the communication and visibility graphs of a group of agents are such that they enable the maintenance of a geometrically rigid shape of their formation then there is a distributed algorithm and communication protocol that enables all their local coordinate systems to take the same orientation.

Proof. If the agents are able to maintain a rigid formation based on visibility (i.e. that they are able to locate other agents in their local coordinate frames) and communication links among themselves, that implies two facts: (1) their communication graph must be connected and (2) each agent can see at least two other agents (see Lemma 2 below). The latter however ensures that each agent is able to unambiguously locate the rigid shape of the formation in their own local coordinate system. This means that they can align their coordinate systems with a frame fixed to the rigid shape of the formation that is shared. □

Note however that the agents' ability to align their local coordinate systems (based on visibility and communications) *does not imply rigidity* of their formation. They may be able to have a variable-geometry formation and still maintain aligned local frames (this goes beyond our scope of this paper and it is the subject of further research).

2.1 Overview of graph rigidity

The concept of graph rigidity used in problems related to formation control is not new to the literature. For instance, the last edition of 2008 Control Systems Magazine was dedicated to this subject. For this paper, however, we are not interested in formation control, and hence rather than holding them to or rearranging the robots to a particular configuration, we want them to share the distances of the neighbours they can see, and with that information establish a *common orientation*.

It is widely accepted that the first contribution in graph rigidity was due to Euler, with a conjecture about polyhedra. After Euler, Cauchy proved that two convex polyhedra with congruent faces are congruent in $\mathbb{R}^3$. Asimow and Roth (Asimow and Roth (1978)) proved that if a rod and pin-joint structure in $\mathbb{R}^3$ defines a convex polygon that is rigid, then all of its faces must be triangular.

Even though the problem was first proposed in $\mathbb{R}^3$, we still don't have sufficient and necessary conditions that guarantee the rigidity for any particular graph based on the number of pairwise distances fixed relative to the number of vertices.

2.2 Rigidity in 2D

In 2D, the set of conditions established by Laman (1970) are both necessary and sufficient. A result on 2D rigidity presented in Mantler and Snoeyink (2004):

Theorem 3. (Laman Conditions for 2D). A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is rigid for dimension 2 if and only if there is a subset $\mathcal{E}'$ of $\mathcal{E}$ such that:

(1) $|\mathcal{E}'| = 2|\mathcal{V}| - 3$, and
(2) for all $\mathcal{E}'' \subset \mathcal{E}'$ where $|V(\mathcal{E}'')| \geq 2$, we have $|\mathcal{E}''| \leq 2|\mathcal{V}(\mathcal{E}'')| - 3$.

Their condition emphasized rigidity as a consequence of the connectivity on the graph in terms of distances known between agents, rather than on the geometry that the connections represent. In this paper, we discuss how for 3D connectivity does not tell the whole story, and how it is possible to present some result instead by looking at geometric properties of the resulting graph as dependent on visibility and communications graphs.

2.3 Rigidity for shared orientation

We study rigidity in 3D only as tool, and use it as an intermediate step in order to establish a uniform orientation for reference frames. In the next section, we discuss some of the current results for rigidity of general polyhedra in 3D. In our main results we will focus on basic formations where the agents are deployed on a circular path of length N, which corresponds to their communication and visibility graphs; and all the information they have available is the location of their two neighbours in their own local coordinate systems. We would thus derive a result that is similar to that from Asimow and Roth, but without the restriction of convexity. We are able to do so because, in addition to relative distances (rod-joint formations), we can exploit the fact that the agents would know the relative location of its neighbours.

3. PROBLEM OF FRAME ALIGNMENT IN 3D

When dealing with structures in 3D, our natural approach is to exploit a potential rigidity on the underlying formation to define a universal reference frame. However, the problem of finding sufficient conditions for rigidity in 3D is still open in terms of the graph of distances known between agents. To provide some background, we present an extension to Laman's conditions, as first presented in Graver et al. (1993).

Definition 4. (Extension of the Laman condition). A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a *Laman graph* in 3D if there is a subset $\mathcal{E}' \subseteq \mathcal{E}$ such that:

(1) $|\mathcal{E}'| = 2|\mathcal{V}| - 6$, and
(2) $\forall \mathcal{E}'' \subseteq \mathcal{E}'$ where $|\mathcal{V}(\mathcal{E}'')| \geq 3$, then $|\mathcal{E}''| \leq 3|V(\mathcal{E}'')| - 6$.

The relevance of this condition is that a similar condition implied rigidity in 2D. Condition (2) (that no subgraph can have too many edges) ensures that each edge contributes to reducing the overall number of degrees of freedom, and is not wasted within a subgraph that is already itself rigid due to its other edges. It was proven in Mantler and Snoeyink (2004) that there are limitations in the number of edges that can be added to subgraphs under Laman's condition. First we introduce the concept of k-vertex connectivity.

Definition 5. (k-vertex connected). A graph $\mathcal{G}$ is said to be k-vertex connected if it is connected and there is a set of k vertices so that by removing them $\mathcal{G}$ becomes disconnected, and there is no set of $k-1$ vertices so that, by removing them, $\mathcal{G}$ becomes disconnected ((West, 2001, p. 149)).

Theorem 6. (Theorem 1 in Mantler and Snoeyink (2004)). A minimal graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that satisfies Laman's condition is at most 5-connected. Furthermore, there are k-connected *flexible* graphs for $k = 3, 4, 5$ that satisfy the Laman condition for 3D.

This means that vertex connectivity alone cannot be used to construct a rigidity test.

3.1 Some basic results

For our purposes this means that the distance-specific degree of connectivity cannot be the *only* information used by the agents in order to make decision on the possibility of a universal orientation for their local reference frames. Here we show that exploiting triangular structures, it is possible to reach such an agreement on the orientation of the reference frame.

Lemma 2. If $\mathcal{V} = \{i, j\}, \mathcal{O} = \emptyset$ and $\mathcal{G}_V = \mathcal{G}_C = \{\mathcal{V}, \{(i,j), (j,i)\}\}$ then agents i and j are unable to align their coordinate systems.

Proof. Suppose that agents i and j are capable of aligning their coordinate frame. Without loss of generality, assume that j, as seen by i, is on position $(0, 0, 1)$. Note that, regardless of the position on which j observes i, any rotation by i, around its z axis, continues to lead to the same information for both agents (this is, their relative positions would not change) as in Figure 1. The same is true for the orientation of the frame of agent j, after rotating it around the line that joins i and j. This ambiguity on their relative orientations leads to the desired result. □

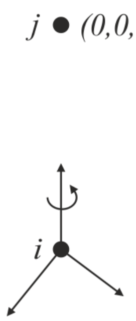

Fig. 1. Two agents alone cannot agree on the orientation of their reference frame by using only their relative positions.

Lemma 3. If $\mathcal{V} = \{i, j\}, \mathcal{O} = \{k\}$ and $\mathcal{G}_c = \{\mathcal{V}, \mathcal{E}_c\} = \{\mathcal{V}, \{(i,j), (j,i)\}\}$, $\mathcal{G}_c = \{\mathcal{V}, \mathcal{E}_c \cup \{(i,k), (j,k)\}\}$ so that i, j, k are not collinear; then agents i and j are able to align their coordinate systems.

Proof. Each agent can define a coordinate system that has its $\boldsymbol{z}$ axis aligned with the line joining i and j; and its $\boldsymbol{y}$ axis to lie on the plane spanned by i, j, k so that the positive $\boldsymbol{z}$ axis direction is from i to j ($i < j$), the $\boldsymbol{x}$ axis is perpendicular to the plane containing the three points, and $\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{z}$ form a right-handed coordinate system, as in Figure 2. The meaning of this definition can be shared by the two agents to form coordinate systems of identical orientation in 3D. □

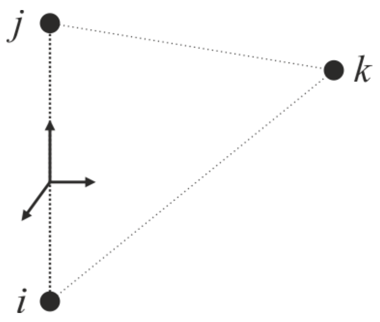

Fig. 2. Two agents alone can agree on the orientation for their reference frames with the aid of a third object.

Lemma 4. If each agent of an agent group $\mathcal{V}$ can see the same object $O \in \mathcal{O}$ and each other and they also have a connected communication graph, then they have the possibility to agree in a shared coordinate system of the same orientation.

Proof. Based on Lemma 3 any two agents $i, j \in \mathcal{V}$ that communicate can have a shared orientation for their coordinate system. Furthermore, agent i can compute the required transformation from its shared frame orientation with agent j to its shared frame orientation with another neighbouring agent k; i.e. agent i can compute $R_{ij,k}$. Similarly for agent j. As any two agents l and m in $\mathcal{V}$ are connected by a communication path, their own coordinate systems transformation can be shared along their communication path. This means that a common orientation for the coordinate system can be agreed; for instance, they can all set the orientation of their local frames to that agreed by the agent with the lowest id, and the agent of lowest id in its neighbouring set. The result follows. □

Remark 1. Note that the previous algorithm is not very scalable. In fact, if new agents are added to the system, or fail due to battery constraints, the previous approach would not converge, unless there is a higher level supervisory stage which would continuously update the number of agents (and their ids). For instance, a distributed counting algorithm like the one in Caicedo-N and Žefran (2010) can allow the agents to keep track of how many agents are active in the formation, and to react when the number changes.

The following corollary provides an initial condition on the ability of a set of agents to agree on a shared orientation for their coordinate systems.

Corollary 1. Let $\mathcal{O} = \emptyset$ and $\mathcal{E} \subset \mathcal{S}$. Suppose that if j and k are both neighbours of i, then i knows whether j, k are also neighbours between themselves. If the communication graph is connected and can be partitioned into triangles, then the agents can agree on a shared orientation of their coordinate systems.

Proof. For each agent i, and each pair of neighbours j, k so that j and k are neighbours between themselves, i, j, k can agree on a local reference frame following the procedure in Lemma 3. For each agent i with more than one pair of neighbours, it would choose the transformation that transforms each of its coordinate frames, into that agreed on the triangle for which its agents have the minimum sum. If there is more than one such a triangle, it will choose which one to transform to at random. Since the graph is connected the information regarding the reference frame and the related transformation is shared as in Lemma 4, the result follows. □

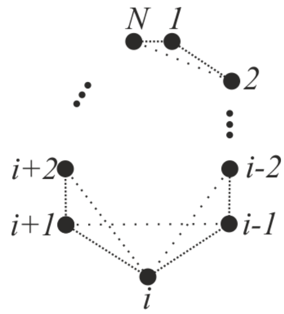

Fig. 3. When the agents are in a circular path, they can obtain the information of all the successive triangles.

Remark 2. Note that this result is consistent with what was established by Asimow and Roth (1978). However, although we don't use the convexity of the formation to derive it, we relied on the property of the agents that they belong to more than one triangle to establish the necessary transformation. This can be done since each agent can place all of its neighbours inside its own reference frame. We discuss this in more detail at the end of the next section.

3.2 Using triangularization

Up to this moment, we have been able to establish that a formation is rigid in 3D when it can be triangularized by a combination of both visibility and communication graphs. A question that naturally arises at this moment is whether an equivalent result can be derived with a less fine tiling of the formation. In the following results, we will first show that if the communication graph defines a cycle of length 7 or higher, then no rigid formation can be defined while in §4 we will examine circular paths of lengths $4, 5, 6$.

Before we proceed with this, we describe the information that each agent can have available. If agent i is neighbour with agent j and k, then agent i not only can estimate the length of the segments $\overline{ij}$ and $\overline{ik}$, but also that of $\overline{jk}$. Henceforth, if agent i is going to exchange information with agent j, it can provide the three lengths of the triangle $\triangle i, j, k$. This information can be propagated throughout the network. So, we assume that each agent knows the following:

(1) The length of the distance between any two neighbours, and
(2) The distance to any neighbour of a neighbour.

We can henceforth think of a *semi-triangularization* of the circular path, as shown in Figure 3 where the dotted lines show the distances known around agent i on the cycle.

It is clear that, without previous agreements on the orientation of the reference frames, this is the most complete set of information they can obtain. One can easily verify

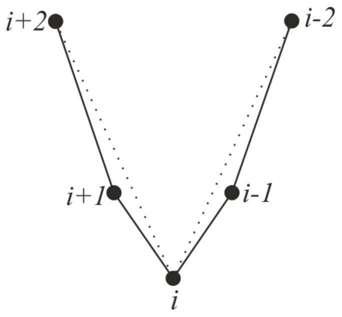

Fig. 4. Information available to agent i via its immediate neighbours.

that unless $N \leq 6$, the formation would have more degrees of freedom than the agents' constraints and hence rigidity would not be attained. As a consequence of this, the agents would not reach an agreement on the orientation of their local reference frames by using only the known distance on their geometric configuration.

Lemma 5. Assume that $\mathcal{O} = \emptyset$ and $\mathcal{E} \subset \mathcal{S}$. If $N \geq 7$ agents define a cycle in the communication graph, then they would not posses enough information to agree on the orientation of the coordinate frame.

Proof. Suppose, without loss of generality, that the agents on the cycle are labelled $1, 2, \cdots, N$, in a clockwise manner. The following discussion is taken modulo N. Each agent i has 3 degrees of freedom (DOF) for its position in $\mathbb{R}^3$. The total number of their DOF in $\mathbb{R}^3$ is $3N$. By agent i observing its neighbours $i-1, i+1$, the location of the remaining $N-3$ reduces the numbers of DOF to characterise the formation to $3(N-3)$. Agents $i-1$ and $i+1$ share with i the location of agents $i-2$ and $i+2$ respectively. The information that now agent i possess is illustrated in Figure 4.

This alone is not sufficient for agent i to place those two additional robots correctly on its own coordinate frame, so agents $i-1$ and $i+1$ also share the distance between i and $i-2$, and i and $i+2$ respectively. This reduces the numbers of degrees of freedom from $3(N-3)$ to $3(N-3)-4$. Proceeding with this sharing, each time agents $i-j$ and $i+j$ provide agent i with the distance information from agent $i-(j+1)$ to $i-j$ and $i-(j-1)$, and from agent $i+(j+1)$ to $i+j$ and $i+(j-1)$ respectively, the degrees of freedom at the formation (for agent i) reduces in 4. Henceforth, if there would be a chance for agent i to completely characterize the formation in a unique way by having access only to the information prescribed, it would be needed that it possesses more constrains than degrees of freedom. This is, $2(N-1) > 3(N-3)$ or, equivalently, $7 > N$ as we claimed.

Alternatively, on could argue that the the number of constraints in the cycle is $2N$ and the total number of DOF of the agenst together needs to be maximum 6, i.e.

$$3N - 2N \leq 6$$

that results $N \leq 6$ for the possibility of rigidity. □

Remark 3. The knowledge of the geometry of each of the triangles $i-1, i, i+1$ define multiple possible cycles $1, 2, \cdots N$ when $4 \leq N \leq 6$. This can be verified by direct inspection. What this means, however, is not that the formation is not rigid, but rather than the resulting solids cannot be derived from each other via a rigid body transformation.

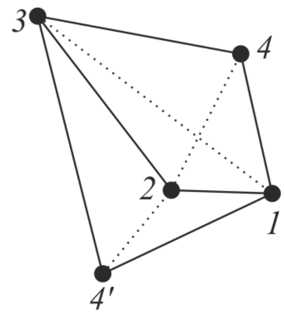

Fig. 5. There are two possible configurations for the 4 agents if only the distances are known.

4. CONDITIONS OF ALIGNMENT

The problem of having multiple polyhedra of the formation, not equivalent to each other by rigid-body transformations despite the completeness of the distance constraints, is an interesting one (there are for instance 2 incongruent tetrahedra for any set of 6 edge lengths, if any) that can be resolved not by looking at rigidity per se, but by trying to solve the reference frame orientation problem. In fact, as discussed earlier, these multiple polyhedra, which are constructed with the information of the triangles $\triangle i-1, i, i+1$ (mudulo N) for $1 \leq i \leq N$ are equivalent via reflexion to a plane. The example for $N = 4$ is shown in figure 5. The induced pyramid $1, 2, 3, 4$ is equivalent, upon reflexion on the triangle $\triangle 1, 2, 3$ of the agent 4 (to become agent $4'$ in the figure). It is clear that the knowledge of the linear segments is not enough to guarantee a unique formation and, henceforth, the orientation of the reference frame cannot be derived directly from the distances alone. However, one might ask at this point whether it is possible to break the ambiguities for the circular paths of length $4 \leq N \leq 6$ by either exploiting the motion capabilities of the agents. We expand on this point next

4.1 Circular paths of length 4, 5, 6

We will explain how could this work by relying on the case $N = 4$. The extension to $N = 5, N = 6$ follows an equivalent approach.

Note that for the agents not to be able to discern between which of the two polyhedra is the one on which they are actually deployed, it is necessary that the following equalities between the lengths hold: $\overline{1,4} = \overline{1,4'}, \overline{3,4} = \overline{3,4'}$ and $\overline{2,4} = \overline{2,4'}$ (equivalent ambiguities would arise by considering reflections on the triangles $\triangle 2, 3, 4$ and $\triangle 1, 3, 4$). If agent 1 is going to make a *small*— change of its position, the ambiguities would be preserved if and only if it were to move on the plane that contains the perpendicular bisector of the segment $\overline{4,4'}$. Any other motion would break the symmetry: the agents would remember where the previous position of agent 1 was, and then compare, after its displacement has placed it into a new location, whether this new location is consistent with the polyhedron $1, 2, 3, 4$ or with $1, 2, 3, 4'$. Once this is known, agent 1 chooses a reference frame for which the axes $\boldsymbol{x} - \boldsymbol{y}$ lie on the triangle $\triangle 1, 2, 3$, the fourth vertex

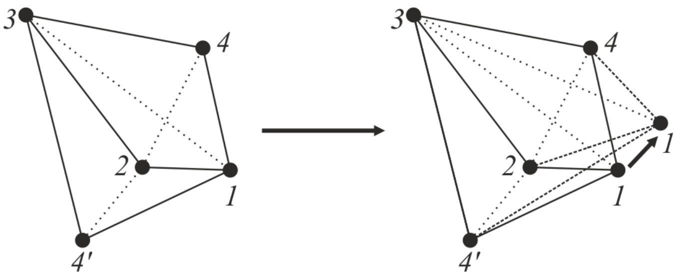

Fig. 6. The motion of the agent with the lowest id breaks the ambiguities.

lies on the positive $\boldsymbol{z}$ axis; and one between the $\boldsymbol{x}$ or $\boldsymbol{y}$ axis lies on the segment $\overline{1,2}$ in such a way that the reference frame is a right-handed frame (Fig.6).

4.2 Main results

We can summarize the previous discussion in the following result. The cases for $N = 5$ and $N = 6$ are equivalent.

Lemma 7. Let $4 \leq N \leq 6$ agents be arranged on a circular path. If each of the agents have access to the information on the shape of each of the triangles $\triangle i-1, i, i+1$ (modulo N), then at least two possible circular paths in $\mathbb{R}^3$ can be constructed. Ambiguity in these paths can be resolved by allowing the agent with the lowest id to perform a random motion in a direction other than that of the symmetry planes.

Combining the previous results, we are now in the position to state the main result of this paper: a condition for rigidity, and hence orientation of the reference frame, in a formation of autonomous agents.

Theorem 8. Consider a formation of N agents deployed in $\mathbb{R}^3$, with $\mathcal{G}_C$ being their communication graph, and $\mathcal{G}_V$ their visibility graph. Given that the agents are capable of identifying the position of their neighbours in their coordinate systems, the following are sufficient and necessary conditions for the possibility of them agreeing on the same orientation of a global frame they would share:

(1) $\mathcal{G}_C$ is connected, and $\mathcal{G}_V$ is 2-connected.
(2) $\mathcal{G}_C$ can be partitioned into circular paths of length less than or equal to 6.

Remark 4. Observe that unlike Laman's conditions (reproduced here in Theorem 3), which rely only on the connectivity of the graph, we also exploit the geometry of the formation to derive the result.

5. CONCLUSIONS

We have presented conditions under which N agents can agree on the orientation of their own local reference frames in a distributed robotic network upon deployment. Under the assumption that a set of N agents is characterized by a circular visibility path, they can agree unambiguously on the orientation of the coordinate frame if and only if the original path was of length less than or equal to 6 by sharing their relative distances. By exploiting connectivity of the communication graph, we have been able to prove that if the visibility graph induced by the agents induce a triangularization, then they can agree on a reference frame. This last result is obtained similarly to that proved by Asimow and Roth in 1978, with the fundamental difference that, by exploiting the ability of the agents to identify the location of its neighbours (and not only their relative distances) we do not rely on the convexity of the polyhedron. Using the motion ability allows to extend the result even further: the agents will be able to agree on the orientation of their local reference frame when the communication graph defines a polyhedron that can be partitioned into cycles of length less than or equal to 6.

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