# Multi-unit Auctions with a Stochastic Number of Asymmetric Bidders

**Ioannis A. Vetsikas**[1] and **Sebastian Stein**[2] and **Nicholas R. Jennings**[2]

**Abstract.** Existing work on auctions assumes that bidders are symmetric in their types — they have the same risk attitude and their valuations are drawn from the same distribution. This is unrealistic in many real-world applications, where highly heterogeneous bidders with different risk attitudes and widely varying valuation distributions commonly compete with each other. Using computational service auctions that are emerging in cloud and grid settings as a motivating example, we examine how an intelligent agent should bid in such multi-unit auctions with asymmetric bidders. Specifically, we describe the equilibrium bidding strategies in three different settings that are distinguished by the levels of uncertainty about the types of other agents. First, we consider a setting with full knowledge about all agents' types, then we consider the case where the types are uncertain, but the number of bidders is known. Finally, we consider the case where both the number of bidders and their types are uncertain. Our experiments show that using the equilibrium strategies derived from our full analysis leads to increased utility (typically $20 - 25\%$) for the participants compared to previous state-of-the-art strategies.

## 1 Introduction

Once confined to specialist domains such as the arts or antiques trades, auctions have now become a pervasive feature of our daily lives. Whether used by governments to sell bonds, by companies to trade commodities or by private individuals to buy second-hand goods online, auctions offer a number of key advantages. In particular, they determine prices dynamically to balance supply and demand, they ensure that goods are allocated to the highest bidders, and, finally, they can be conducted electronically over the Internet to connect buyers and sellers on a global scale.

One particular application example of auctions that has started to emerge recently is in the area of computational services (exemplified by cloud or grid computing). Increasingly, companies rent out spare computational resources to paying customers [3]. This allows them to profit from otherwise idle hardware, while customers have the flexibility of temporarily using expensive resources for demanding computational tasks without having to invest in expensive hardware themselves. Although much research in this area has so far concentrated on models with fixed prices [14], auctions are increasingly emerging as a suitable mechanism for balancing supply and demand in these settings [2, 11], and they are starting to emerge in real-world cloud systems, such as Amazon's EC2 spot instances[3] and the SpotCloud[4] platform. However, as companies are often interested in dynamically procuring services on demand without human intervention, this raises the research question of how to build intelligent software components, or agents, that bid automatically in these auctions.

To answer this question, we can turn to a considerable body of work that has investigated optimal bidding strategies in auctions. Typically, research in this area has used techniques from game theory to model the behavior of participants and to derive appropriate strategies [6]. However, such work has usually made a number of simplifying assumptions. In particular, it has been assumed that bidders are symmetric (i.e., that the valuations for the good they bid on are drawn from the same prior distribution and that they share the same utility model) and that the number of bidders is known a priori. These assumptions do not hold in the domain of computational services, where bidders can be highly asymmetric [1]. In particular, their valuations can be fundamentally different, depending on how critical the services are to their business needs. Furthermore, risk attitudes between different companies can vary significantly. As an example, a large company with a considerable reputation and many stakeholders may be more risk averse than a small startup company; and these differences could be further amplified across different industry sectors. Finally, because of the global nature of these auctions, the exact number of bidders is not generally known in advance.

Some existing research has considered some of these issues separately. For example, equilibria for auctions with asymmetric bidders with different prior distributions from which their valuations are drawn have been computed in [7, 9], and an experimental evaluation is conducted in [4]. However, these assume that the distributions and the number of bidders are common knowledge. Other work has considered different utility models, including risk aversion [6, 8] which apply only to symmetric settings, i.e., where all agents share the same utility model. Furthermore, the question of auctions with stochastic numbers of bidders has been examined in [5, 10]. A further limitation is the fact that the analysis in all the aforementioned papers is conducted for single-unit auctions.

To address these shortcomings, we examine how an intelligent agent should bid in multi-unit service auctions with asymmetric bidders, where each bidder has a different type (i.e., a particular risk attitude and distribution from which its valuation is drawn). More specifically, we develop equilibrium bidding strategies for three realistic settings that vary in their respective levels of uncertainty about the bidders. We start by assuming the number of bidders is known and then first consider the case where the types of all agents are public knowledge. Second, we consider the case where types are private knowledge, and, third, we extend our work to settings with an uncertain (i.e., stochastic) number of participants.

While we use computational services as a motivating example

[1] IIT, NCSR Demokritos, Greece, email: ivetsikas@iit.demokritos.gr
[2] ECS, University of Southampton, UK, email: {ss2,nrj}@ecs.soton.ac.uk

[3] `http://aws.amazon.com/ec2/spot-instances`
[4] `http://spotcloud.com`

throughout this paper, we stress that our work applies more broadly to any setting where bidders are highly asymmetric in their valuations and risk attitudes. This could include real estate, spectrum or government bond auctions.

The remainder of this paper is organised as follows. In the next section, we formally present the model and the notation that will be used in this paper. Then, we derive the systems of differential equations that characterize the Bayes-Nash equilibria in the three settings outlined above. Afterwards, we present experiments showing that our analysis leads to significantly increased utility for the participants compared to state-of-the-art strategies. Finally, we conclude.

## 2 Model and Notation

In this section we formally describe the auction model and notation used in our analysis. We study sealed-bid, uniform-price, multi-unit auctions, as these are widely used and because one of the key commercial cloud offerings, Amazon's EC2 spot instance auction, is based on a similar model. We use the standard assumptions regarding the format of the utility function that each agent maximizes and also assume that the losers of the auction do not pay anything.

We will compute Bayes-Nash equilibria for sealed-bid auctions where $m \geq 1$ identical items, or services, are being sold; these equilibria will be defined by a set of strategies $g_{\alpha_i}(v)$, which map the agents' valuations $v_i$ to the bids $b_i$ submitted in the auction. Since, unlike in the vast majority of related work, the bidders are asymmetric, these strategies are parameterized by a parameter $\alpha_i$, which will indicate the model of agent $i$, i.e., its risk attitude and type of valuations. Thus, we assume that two agents will use the same bidding strategy, if they have the same model (same parameter $\alpha_i$). The final price, which is paid by all winning bidders, is determined by the $m^{th}$ price rule, according to which the top $m$ bidders win one item each at a price equal to the $m^{th}$ highest (last winning) bid respectively.

We assume that $N$ bidders (where $N \geq m$) participate in the auction and each has a private valuation (utility) $v_i$ for acquiring any one[5] of the traded items, which is known only to itself; these valuations are assumed to be independent and drawn from a distribution with cumulative distribution function (cdf) $F_{\alpha_i}(v)$, which depends on the bidder's model $\alpha_i$. Furthermore, we assume that $F_{\alpha_i}(v)$ has support in $[v_i^L, v_i^H]$, which means that $\forall v \notin [v_i^L, v_i^H]$ it is $F'_{\alpha_i}(v) = 0$.The agents have varying risk attitudes. The possible risk attitudes belong to a family of utility functions $u_{\alpha_i}()$, which are characterized by the type (model) $\alpha_i$ of each agent. Thus, we assume that the utility $U_i$ that each agent tries to maximize is equal to:

$$U_i = \begin{cases} u_{\alpha_i}(v_i - p_{closing}) & \text{if agent } i \text{ wins,} \\ u_{\alpha_i}(0) & \text{if agent } i \text{ loses.} \end{cases}$$

where $p_{closing}$ is the closing price of the auction.

The family of utility functions $u_\alpha(x)$ used most commonly in economics is the Constant Relative Risk Aversion (CRRA) family, $u_\alpha(x) = x^\alpha$, $\alpha \in (0, 1)$, which characterizes risk-averse bidders and is defined for non-negative profit $x \geq 0$. We are thus able to use it in this paper, as there is never a need to bid higher than one's valuation, hence the profit will always be non-negative. To handle risk-seeking bidders as well, we extend the model and use values of $\alpha > 1$; in this case the utility function will be convex and therefore will characterize such bidders.

For this model, we examine the following cases:

1. The models of all participants are known, i.e., each agent knows the parameters $\alpha_j$ of its opponents. A prominent example of this type of setting is a spectrum auction, where bidders will generally be aware of each other and will have information about each other's risk attitudes and valuation distributions.
2. Each agent knows its own model, but is uncertain about its opponents' parameters $\alpha_j$. Specifically, the agents know the prior distribution $h(\alpha)$ from which each opponent's parameter $\alpha$ is drawn, which is the probability that each participant is of a particular type $\alpha$ (we assume this distribution to be discrete, although the results can be extended to continuous distributions). This setting covers realistic auctions, such as real estate auctions in which bidders have little prior knowledge about their competitors, or auctions for specialised cloud services, where the bidders are known to each other, but their risk attitudes and valuations might vary depending on their current projects.
3. Both the opponents' models and the number of bidders $N$ are uncertain. As above, the distribution $h(\alpha)$ is known as well as a distribution $p(N)$, which details the probability that exactly $N$ bidders will participate in the auction. This case captures most computational service auctions, which typically have considerable uncertainty both in the number of bidders and their types.

Before considering each of these cases in turn, we introduce some additional notation that we will use in the proofs:

$$\Phi_k(x) = \sum_{i=0}^{k-1} C(N-1, i) x^{N-1-i} (1-x)^i \tag{1}$$

$$\Delta\Phi_m(x) = \Phi_m(x) - \Phi_{m-1}(x) = C(N-1, m-1) x^{N-m} (1-x)^{m-1} \tag{2}$$

where $C(n, k)$ is the total number of possible combinations of $k$ items chosen from $n$. Now, if $Z(x)$ is the probability distribution of any opponent's bid $b_j$, i.e., $Z(x) = Prob[b_j \leq x]$, and $B^{(k)}$ is the $k^{th}$ order statistic of these bids, then the distribution of $B^{(k)}$ is: [13]

$$Prob[B^{(k)} \leq x] = \Phi_k(Z(x)) \tag{3}$$

$\forall N, m$, such that $N \geq m$, the following equation holds: [15]

$$\Phi'_m(x) = (N-m)\big(\Phi_m(x) - \Phi_{m-1}(x)\big)\frac{1}{x} \tag{4}$$

Before proceeding to detail the theoretical analysis of $m^{th}$ price auctions in the next sections, we present here the dominant strategy for the $(m+1)^{th}$ price variant. It is known from auction theory that it is a dominant strategy to bid truthfully [6], and this fact does not depend on the number of participating bidders, nor on the risk attitudes and valuation distributions of the participants:

**Fact 1.** *In the case of an $(m+1)^{th}$ price sealed-bid auction with $N$ participating bidders, in which each bidder $i$ is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to $v_i$, and has a risk attitude described by utility function $u_{\alpha_i}()$, it is a (weakly) dominant strategy to bid truthfully: $b_i = v_i$.*

## 3 Known Opponent Models

In this section, we compute the equilibrium strategies for auctions where the participating bidders have asymmetric valuations and risk attitudes, and the models of all opponents (i.e., their valuation distributions and risk attitudes) are common knowledge to all participants. We first give a definition of a term used in the theorem.

[5] We make the assumption that each bidder is interested in exactly one item, which is the usual assumption made as multi-unit demand is an open problem. Thus, in cloud applications, our work is most relevant to high-value computational services, such as, e.g., the exclusive use of a supercomputer over a 24-hour period, or a specialised high-performance cluster for graphics rendering.

**Definition 1.** *Given a set $S$ (with all its elements being unique), let us define the $k$-subset $S^{(k)}$ of $S$ to be the subset of its powerset $2^S$ whose elements have cardinality $k$ (with $k \leq |S|$). More formally:*

$$S^{(k)} = \{s \in 2^S : |s| = k\}$$

*For the specific case of the set $S = \{1, \ldots, m\}$, let us define:*

$$P_{k,m} = \{1, \ldots, m\}^{(k)}$$

*which is the set containing all the possible ways of selecting $k$ different numbers out of the set of numbers 1 through $m$. Finally, we define the extensions of this definition:*

$$P_{k,m}^{-\{i\}} = (\{1, \ldots, m\} - \{i\})^{(k)}$$

$$P_{k,m}^{-\{i,j\}} = (\{1, \ldots, m\} - \{i, j\})^{(k)}$$

*which are the $k$-subset of all the numbers 1 through $m$ without counting any subsets containing $i$ (and $i, j$ respectively).*

Note that $P_{k,m} = \{\emptyset\}$ for $k = 0$, which is a set containing with one element, the empty set, whereas $P_{k,m} = \emptyset$ for $k < 0$, which is the empty set.

**Theorem 1.** *In the case of an $m^{th}$ price sealed-bid auction with $N$ participating bidders, in which each bidder $i$ is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to $v_i$, which is drawn from $F_{\alpha_i}(v)$, with support $[v_i^L, v_i^H]$ (where $v_i^L = v_i, \forall i$), and has a risk attitude described by utility function $u_{\alpha_i}()$, both of which describe its model $\alpha_i$ and the models $\alpha_i$ of all bidders are common knowledge, then strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \zeta'_{\alpha_j}(x) F'_{\alpha_j}(\zeta_{\alpha_j}(x))) \sum_{s \in P_{m-1,N}^{-\{i,j\}}} \Big( \prod_{\substack{\mu \notin s \\ \mu \neq i,j}} F_{\alpha_\mu}(\zeta_{\alpha_\mu}(x)) \prod_{\mu \in s} (1 - F_{\alpha_\mu}(\zeta_{\alpha_\mu}(x))) \Big) =$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x) - x)}{(u_{\alpha_i}(\zeta_{\alpha_i}(x) - x) - u_{\alpha_i}(0))} \sum_{s \in P_{m-1,N}^{-\{i\}}} \Big( \prod_{\substack{j \notin s \\ j \neq i}} F_{\alpha_j}(\zeta_{\alpha_j}(x)) \prod_{j \in s} (1 - F_{\alpha_j}(\zeta_{\alpha_j}(x))) \Big) \tag{5}$$

*with boundary conditions: $\zeta_{\alpha_i}(v^L) = v^L$ for all $i$.*

*Proof.* Due to space limitations, we omit the proof. It can be found in our workshop paper [17], where initial work was presented. □

## 4 Unknown Opponent Models

In this section, we assume that each agent has uncertainty not only for the opponents' valuations, but also for their models (i.e., risk attitudes and distributions of valuations). The possible risk attitudes and distributions of valuations belong to a family of functions, which are characterized by an one dimensional parameter $\alpha$, which is drawn from a known probability distribution ($h$). We therefore assume that each agent $i$ knows its own valuation $v_i$, risk attitude function $u_{\alpha_i}()$ and the distributions $F_{\alpha_i}(v)$, as well as the distribution $h(\alpha)$ from which the models of the opponents, meaning the risk attitude functions $u_\alpha()$ and distributions $F_\alpha(v)$, are drawn. We assume that there are $\lambda$ possible models, which are characterized by parameters $\alpha = \alpha_1, \ldots, \alpha_\lambda$. We initially present the system of equations that characterize the equilibrium and then show how to solve them.

**Theorem 2.** *In the case of an $m^{th}$ price sealed-bid auction with $N$ participating bidders, in which each bidder $i$ is interested in purchasing one unit of the good for sale with inherent utility (valuation) for that item equal to $v_i$, which is drawn from $F_{\alpha_i}(v)$, and has a risk attitude described by utility function $u_{\alpha_i}()$, both of which describe its model $\alpha_i$ (where $\alpha_i$ are i.i.d. random variables drawn from distribution $h(\alpha)$), strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations:*

$$\forall x, \alpha_i : (N - m) \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F'_\alpha(\zeta_\alpha(x)) \zeta'_\alpha(x) h(\alpha) = \tag{6}$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x) - x)}{u_{\alpha_i}(\zeta_{\alpha_i}(x) - x) - u_{\alpha_i}(0)} \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(\zeta_\alpha(x)) h(\alpha)$$

*with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L$ for all $i$ such that $v_i^L = \min_j \{v_j^L\}$. There are $\lambda$ possible bidder models characterized by parameter $\alpha = \alpha_1, \ldots, \alpha_\lambda$.*

*Proof.* The distribution from which the bid $b_j$ of an opponent with model $\alpha_j$ is drawn has cdf: $Prob[b_j \leq x | \alpha_j] = F_{\alpha_j}(g_{\alpha_j}^{-1}(x))$. Therefore, using Bayes' rule we compute this probability for any possible value of $\alpha_j$:

$$Prob[b_j \leq x] = \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(g_\alpha^{-1}(x)) h(\alpha) \tag{7}$$

The distribution of the $k^{th}$ highest opponent bid $B^{(k)}$, as there are $(N - 1)$ opponents, is:

$$Prob[B^{(k)} \leq x] = \Phi_k \Big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(g_\alpha^{-1}(x)) h(\alpha) \Big) \tag{8}$$

where $\Phi_k(x)$ is given by Equation 1.

We can now analyze the expected profit of bidder $i$. Let $b_i$ be the bid that it places in the auction. We distinguish the following cases:
(i) If $b_i < B^{(m)}$, then bidder $i$ is outbid and does not win any items, therefore its utility is $u_i = u_{\alpha_i}(0)$.
(ii) If $B^{(m)} \leq b_i \leq B^{(m-1)}$, then bidder $i$ has placed the last winning bid. Thus, the payment equals its bid and its utility is $u_i = u_{\alpha_i}(v_i - b_i)$. The probability of this case happening is: $Prob[B^{(m)} \leq b_i \leq B^{(m-1)}] = \Delta\Phi_m \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(g_\alpha^{-1}(b_i)) h(\alpha) \big)$.
(iii) If $B^{(m-1)} < b_i$, then bidder $i$ is a winner, the payment is equal to bid $B^{(m-1)}$ and its utility is $u_i = u_{\alpha_i}(v_i - B^{(m-1)})$. Note that: $Prob[B^{(m-1)} \leq \omega] = \Phi_{m-1} \Big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(g_\alpha^{-1}(\omega)) h(\alpha) \Big)$.

The expected utility of bidder $i$, who places bid $b_i$, is:

$$EU_i(b_i) = u_{\alpha_i}(0) \Big( 1 - \Phi_m \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha) \big) \Big) \tag{9}$$

$$+ u_{\alpha_i}(v_i - b_i) \Delta\Phi_m \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha) \big)$$

$$+ \int_0^{b_i} u_{\alpha_i}(v_i - \omega) \frac{d}{d\omega} \Big( \Phi_{m-1} \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(\omega)) h(\alpha) \big) \Big) d\omega$$

The bid which maximizes this expected utility is found by setting: $\frac{dEU_i}{db_i} = 0$. This becomes:

$$(u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0)) \frac{d}{db_i} \Phi_m \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha) \big)$$

$$= u'_{\alpha_i}(v_i - b_i) \Delta\Phi_m \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha) \big) \tag{10}$$

Thus, using Equation 4 to simplify Equation 10, we derive:

$$(N-m) \frac{\frac{d}{db_i} \big( \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha) \big)}{\sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F(g_\alpha^{-1}(b_i)) h(\alpha)} = \frac{u'_{\alpha_i}(v_i - b_i)}{u_{\alpha_i}(v_i - b_i) - u_{\alpha_i}(0)}$$

This value $b_i$ is equal to $b_i = g_{\alpha_i}(v_i)$, since it maximizes the expected utility $EU_i(b_i)$. Using this substitution, we derive the system of differential equations:

$$\forall v_i, \alpha_i : \ (N - m) \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} \frac{F'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))}{g'_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i)))} h(\alpha) = \tag{11}$$

$$\frac{u'_{\alpha_i}(v_i - g_{\alpha_i}(v_i))}{u_{\alpha_i}(v_i - g_{\alpha_i}(v_i)) - u_{\alpha_i}(0)} \sum_{\alpha = \alpha_1, \ldots, \alpha_\lambda} F_\alpha(g_\alpha^{-1}(g_{\alpha_i}(v_i))) h(\alpha)$$

for all possible values of $v_i, \alpha_i$. The boundary conditions come from the fact that a bidder with the lowest possible valuation that any bidder can have $v_i = v_i^L$ will always bid $b_i = v_i^L$.

Now, to simplify these equations, we make the following substitutions:

(i) As the equations hold for all $\forall v_i, \alpha_i$, therefore, if we set a new variable $x = g_{\alpha_i}(v_i)$, which takes values in $x \in [g_{\alpha_i}(v_i^L), g_{\alpha_i}(v_i^H)]$, we transform the equations to the following:

$$\forall x, \alpha_i : (N-m) \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} \frac{F'_\alpha(g_\alpha^{-1}(x))}{g'_\alpha(g_\alpha^{-1}(x))} h(\alpha) = \tag{12}$$

$$\frac{u'_{\alpha_i}(g_{\alpha_i}^{-1}(x) - x)}{u_{\alpha_i}(g_{\alpha_i}^{-1}(x) - x) - u_{\alpha_i}(0)} \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(g_\alpha^{-1}(x)) h(\alpha)$$

(ii) By setting $\zeta_{\alpha_i}()$ to be the inverse function of $g_{\alpha_i}()$, the equation becomes the system of equations 6. □

**Computing the Equilibrium Strategies** The equations 6 seem quite complex. Thus, we show in this section how to solve them. We assume that $\alpha_i$ are ordered based on the value of $v_i^L$, meaning that we order them so that $v_1^L \leq \ldots \leq v_\lambda^L$. This assumption is crucial for the following steps to work:

(i) In order for this system to have a solution, it must be:

$$\frac{u'_{\alpha_1}(\zeta_{\alpha_1}(x) - x)}{u_{\alpha_1}(\zeta_{\alpha_1}(x) - x) - u_{\alpha_i}(0)} = \ldots = \frac{u'_{\alpha_\lambda}(\zeta_{\alpha_\lambda}(x) - x)}{u_{\alpha_\lambda}(\zeta_{\alpha_\lambda}(x) - x) - u_{\alpha_\lambda}(0)} \tag{13}$$

This gives $(\lambda - 1)$ independent equations; differentiating each one of these gives us the following:

$$\zeta'_{\alpha_i}(x) = 1 + (\zeta'_{\alpha_1}(x) - 1) \cdot \tag{14}$$

$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x)-x)u'_{\alpha_1}(\zeta_{\alpha_1}(x)-x) - u''_{\alpha_1}(\zeta_{\alpha_1}(x)-x)(u_{\alpha_i}(\zeta_{\alpha_i}(x)-x) - u_{\alpha_i}(0))}{u'_{\alpha_1}(\zeta_{\alpha_1}(x)-x)u'_{\alpha_i}(\zeta_{\alpha_i}(x)-x) - u''_{\alpha_i}(\zeta_{\alpha_i}(x)-x)(u_{\alpha_1}(\zeta_{\alpha_1}(z)-z) - u_{\alpha_1}(0))}$$

which is used to substitute all $\zeta'_{\alpha_i}()$ with terms containing only $\zeta'_{\alpha_1}()$ in Equation 13. Thus, we derive the differential equation: $\zeta'_{\alpha_1}()$ is equal to a function of $\zeta_{\alpha_i}(), \forall i$, where $\zeta_{\alpha_i}()$ can be computed from $\zeta_{\alpha_1}()$ using Equation 13. This is solved by using the standard Runge-Kutta method, whose algorithm is presented in chapter 17 of [12], with one modification: the values of $\zeta_{\alpha_i}(), i = 2, \ldots, \lambda$ are computed at each step from the values of $\zeta_{\alpha_1}()$ solving Equation 13 using the Bisection Method; see chapter 9 of [12] for this algorithm.

(ii) Because in step 1, $x$ is defined for $x \in [g_{\alpha_i}(v_i^L), g_{\alpha_i}(v_i^H)]$, we need to be careful when $\zeta_{\alpha_i}(x) < v_i^L$ or $\zeta_{\alpha_i}(x) > v_i^H$ for any $i$. For such values, it is $F(\zeta_{\alpha_i}(x)) = 0$ and $F(\zeta_{\alpha_i}(x)) = 1$ respectively and also $F'(\zeta_{\alpha_i}(x)) = 0$. When performing the simplification of the previous step, we need to keep in mind this fact and that the equations 13 only hold for values of $x$ such that $\zeta_{\alpha_i}(x) \in [v_i^L, v_i^H]$.

**Example 1.** *$N = 3$ bidders and $m = 2$ items for sale*
*There are two possible models of bidders using the CRRA utility function $u_\alpha(x) = x^\alpha$, one where $\alpha = 1$ (risk-neutral bidder) and another where $\alpha = 0.5$ (risk-averse), both with probability $50\%$. Both types have valuations drawn from the uniform distribution $U[0, 1]$.*
*In this example we have the following system of equations (obtained from equations 13 and 6 by setting $u_\alpha(x) = x^\alpha$ for $\alpha_i = 0.5, 1$ and probabilities $h(0.5) = h(1) = 0.5$):*

$$\frac{1}{\zeta_1(x) - x} = \frac{0.5}{\zeta_{0.5}(x) - x} \tag{16}$$

$$\zeta'_1(x) + \zeta'_{0.5}(x) = \frac{0.5}{\zeta_{0.5}(x) - x}(\zeta_{0.5}(x) + \zeta_1(x)) \tag{17}$$

*We present the equilibrium strategies in figure 1 (left). It is interesting to note that the strategy for each asymmetric risk-averse bidder is, in this example, identical to the case when all its opponents are equally risk-averse (the symmetric bidder case). However, when the valuation is high enough that the risk-neutral opponents would never outbid the risk-averse bidders, the latter increase their bids at a much lower rate as the valuation increases. A similar effect is true for the risk-seeking bidders as well. In fact, we can prove this observation, for cases of bidders with identical valuation distribution functions.*

**Example 2.** *$N = 3$ bidders and $m = 2$ items for sale*
*Two possible bidder types, all using the CRRA utility function $u_\alpha(x) = x^\alpha$ (each type with probability $50\%$):*
*Bidder type 1: Risk-neutral ($\alpha_1 = 1$) with valuations drawn from $F_{\alpha_1}(x) = x^2$, where $x \in [0, 1]$.*
*Bidder type 2: Risk-averse ($\alpha_2 = \frac{1}{2}$) with valuations from $U[0, 1]$.*
*We present the equilibrium strategies in figure 1 (middle). We also present in the same figure, the default strategy of the case when the bidders are symmetric, i.e., the strategy when all bidders are either of type 1 or all of type 2: both of these default cases have the same bidding strategy $g(v) = \frac{2v}{3}$. Essentially, in both cases the bids are higher than those when all bidders are risk-neutral with uniform $U[0, 1]$ valuations, because, (a) the risk-averse bidders bid higher than in the case of only risk-neutral ones (all with uniform valuations) due to the desire to increase the probability of winning, (b) the risk-neutral bidder with valuations drawn from $F_{\alpha_1}(x)$, bids higher than in the case of uniform priors due to the higher valuations of the opponents. Now, in the case that there is uncertainty about the types of bidders participating in the auction, we observe that the strategy of the risk-averse bidders is to bid higher than the default strategy, whereas the risk-neutral ones bid lower. This is a result of the fact that the mixture of agents has changed and now the risk-averse bidders face opponents with higher valuations, which forces higher bids, whereas the opposite is the case for the risk-neutral ones and thus they lower their bids.*

Theorem 2 states the boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L$ for all $i$ such that $v_i^L = \min_j\{v_j^L\}$. However, what is the boundary condition for the remaining bidder types? Meaning what is the value of $g(v_i^L)$ for all $i : v_i^L > \min_j\{v_j^L\}$? The equilibrium strategies in this case are computed using the following algorithm:

Set $S = \arg\min_j\{v_j^L\}$.[6]
Set as boundary condition $g_{\alpha_j}(v_j^L) = v_j^L, \forall j \in S$.
**while** $|S| < \lambda$ **do**
  Solve the system of equations 6 setting $F_{\alpha_j}(x) = 0, \forall j \notin S$.
  $\forall j \notin S$: compute the best bid $b_j$ for bidder type $j$ with valuation $v_j^L$, based on the strategies computed in the previous step.
  $S' = \arg\min_j\{b_j\}$.
  Set as boundary condition: $g_{\alpha_j}(v_j^L) = b_j, \forall j \in S'$.
  $S = S \bigcup S'$.
**end while**
Solve the system of equations 6 using the boundary conditions found by the loop of this algorithm.

How this algorithm works is exemplified in the following:

**Example 3.** *$N = 3$ bidders and $m = 2$ items for sale*
*Three possible bidder types, all using the CRRA utility function $u_\alpha(x) = x^\alpha$ (each type with probability $\frac{1}{3}$):*
*Bidder type 1: Risk-neutral ($\alpha_1 = 1$) with valuations from $U[0, 1]$.*
*Bidder type 2: Very risk-averse ($\alpha_2 \to 0$) with valuations from $U[\frac{2}{3}, \frac{4}{3}]$.*
*Bidder type 3: Risk-seeking ($\alpha_3 = 2$) with valuations from $U[1, 2]$.*
*We present the equilibrium strategies in figure 1 (right). Initially, only the boundary condition for the first bidder type is set, i.e., $g_{\alpha_1}(0) = 0$. It is $S = \{1\}$. The bidders belonging to the remaining types, will not bid for the lowest valuations $F_{\alpha_2}(x) = F_{\alpha_3}(x) = 0$. So we compute the bidding strategy under these conditions, and find that $g_{\alpha_1}(v) = \frac{v}{2}$. Now, the optimal bid when bidder type 2 has valuation $v_3^L = \frac{2}{3}$ is $b_2 = \frac{2}{3}$ and the optimal bid when bidder type 3 has valuation $v_3^L = 1$ is $b_3 = \frac{1}{3}$. Thus, $S' = \{3\}$ and $S = \{1, 3\}$. Now, we compute the bidding strategies under these conditions. We*

[6] arg min returns a set of arguments instead of just one in the case that multiple valuations are equal and the minimum ones.

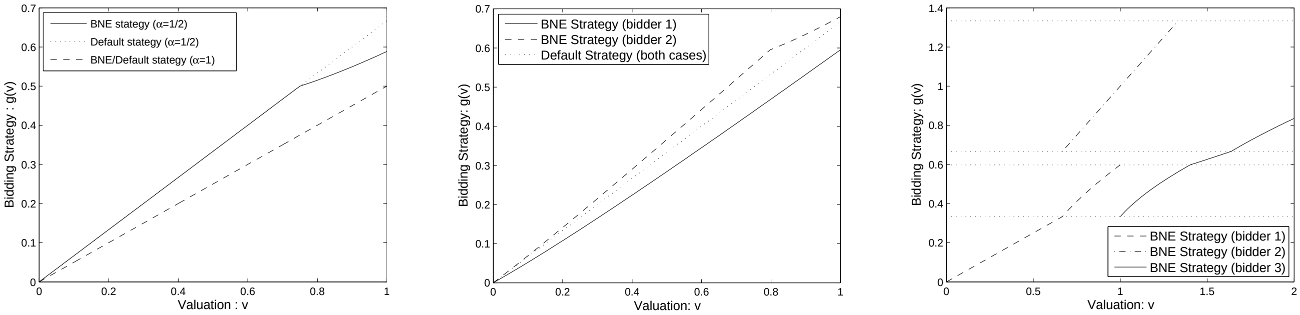


**Figure 1.** Equilibrium strategies $g(v)$ for Examples 1, 2 and 3. Default strategies (i.e., strategies for symmetric cases) are also presented for Examples 1 and 2.

$$\forall x, \alpha_i : \sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F'(\zeta_\alpha(x))\zeta'_\alpha(x)h(\alpha)\sum_{n=1}^{\infty}(n-m)p(n)\Delta\Phi_{m,n}\Big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(\zeta_\alpha(x))h(\alpha)\Big) =$$
$$\frac{u'_{\alpha_i}(\zeta_{\alpha_i}(x)-x)}{u_{\alpha_i}(\zeta_{\alpha_i}(x)-x)-u_{\alpha_i}(0)}\sum_{n=1}^{\infty}p(n)\Delta\Phi_{m,n}\Big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(\zeta_\alpha(x))h(\alpha)\Big)\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda} F_\alpha(\zeta_\alpha(x))h(\alpha) \tag{15}$$

**Figure 2.** System of differential equations characterizing the equilibria of Theorem 3.

*find that $g_{\alpha_1}(1) = .5977$, so up to that point the computed strategies are valid. After that point, we compute the equilibrium bidding strategies for $F_{\alpha_1}(x) = 1$ and $F_{\alpha_3}(x) = 0$. Under these condition, the optimal bid when bidder type 2 has valuation $v_3^L = \frac{2}{3}$ is $b_2 = \frac{2}{3}$, so $S' = \{2\}$ and $S = \{1, 2, 3\}$. Now all the bidding strategies are computed and this produces the bidding strategies of figure 1 (right).*

## 5 Stochastic Number of Bidders

In this section we examine the same setting as in the previous section, with the difference that not only the types of the opponents participating in the auction are not known but their total number is not known a priori either. The participating bidders know instead that the total number $N$ of bidders is given by distribution $p(N)$, where $p(N) = 0, N < 2$.

**Theorem 3.** *Consider the same setting as that of theorem 2, with the difference that the number $N$ of bidders participating is not known: the distribution $p(N)$ gives the probabilities of $N$ bidders participating in the auction. Then, strategy $g_{\alpha_i}(v_i)$ constitutes a Bayes-Nash equilibrium, where $\zeta_\alpha(x) = g_\alpha^{-1}(x)$ is the solution of the system of differential equations presented in figure 2, with boundary conditions: $g_{\alpha_i}(v_i^L) = v_i^L$ for all $i$ such that $v_i^L = \min_j\{v_j^L\}$. There are $\lambda$ possible bidder models characterized by parameter $\alpha = \alpha_1, \ldots, \alpha_\lambda$.*

*Proof.* The expected utility of bidder $i$, who places bid $b_i$, is:

$$EU_i(b_i) = \sum_{n=1}^{\infty} p(n)EU_i(b_i|n)$$

where $EU_i(b_i|n)$ the expected utility of bidder $i$ when the number of opponents is $n$ is given by Equation 9. This equation then becomes:

$$\begin{aligned}EU_i(b_i) =& u_{\alpha_i}(0)\Big(1-\sum_{n=1}^{\infty}p(n)\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big)\Big)\\ &+u_{\alpha_i}(v_i-b_i)\sum_{n=1}^{\infty}p(n)\Delta\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big)\\ &+\int_0^{b_i}u_{\alpha_i}(v_i-\omega)\sum_{n=1}^{\infty}p(n)\frac{d}{d\omega}\Big(\Phi_{m-1,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(\omega))h(\alpha)\big)\Big)d\omega\end{aligned} \tag{18}$$

The bid which maximizes this expected utility, is found by setting: $\frac{dEU_i}{db_i} = 0$. This becomes:

$$(u_{\alpha_i}(v_i-b_i)-u_{\alpha_i}(0))\sum_{n=1}^{\infty}p(n)\frac{d}{db_i}\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big)$$

$$= u'_{\alpha_i}(v_i-b_i)\sum_{n=1}^{\infty}p(n)\Delta\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big) \tag{19}$$

Using Equation 4 to simplify Equation 19, we derive:

$$\frac{\frac{d\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)}{db_i}}{\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)}\sum_{n=1}^{\infty}(n-m)p(n)\Delta\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big)$$
$$= \frac{u'_{\alpha_i}(v_i-b_i)}{u_{\alpha_i}(v_i-b_i)-u_{\alpha_i}(0)}\sum_{n=1}^{\infty}p(n)\Delta\Phi_{m,n}\big(\sum_{\alpha=\alpha_1,\ldots,\alpha_\lambda}F_\alpha(g_\alpha^{-1}(b_i))h(\alpha)\big) \tag{20}$$

This value of $b_i$ that maximizes the expected utility $EU_i(b_i)$ is equal to $b_i = g_{\alpha_i}(v_i)$. We make this substitution and also set a new variable $x = g_{\alpha_i}(v_i)$, which takes values in $x \in [g_{\alpha_i}(v_i^L), g_{\alpha_i}(v_i^H)]$. Now, by setting $\zeta_{\alpha_i}()$ to be the inverse function of $g_{\alpha_i}()$, the equations becomes the system of equations 15. □

## 6 Experiments

In this section, we experimentally validate our analysis even for cases where many bidders do not use the equilibrium strategies and therefore there are no theoretical guarantees. We consider the following case, which is inspired from our motivating example of using computational service auctions that are emerging in cloud and grid settings. The bidders are bidding for $m = 3$ identical time slots of high-value computational services. Some of the bidders are established, risk-averse companies with high value jobs to execute and some others are risk-seeking companies trying to make a larger profit and which can have a more varying spectrum of valuations. So, the two possible bidder types (each type with probability 50%), which both use the CRRA utility function $u_\alpha(x) = x^\alpha$, are:
Bidder type 1: Risk-averse ($\alpha_1 = \frac{1}{2}$) with valuations from $U[\frac{1}{2}, 1]$.
Bidder type 2: Risk-seeking ($\alpha_2 = \frac{3}{2}$) with valuations from $U[0, 1]$.
The total number of participating bidders is not known a-priori, but rather it is known that $N = 4 \ldots 8$ bidders will participate (each case occuring with 20% probability).

Using theorem 3 and the algorithm used in Example 3, we compute the equilibrium strategy for this case, which is presented in figure 3 (left). In this figure, we also present the strategies that we compare against, which are derived from the pre-existing state-of-the-art equilibrium analysis [5, 7]; we extended these somewhat in order to account for multiple units sold and [7] was extended to cover asymmetry in risk attitudes. In either case, we wanted the best possible

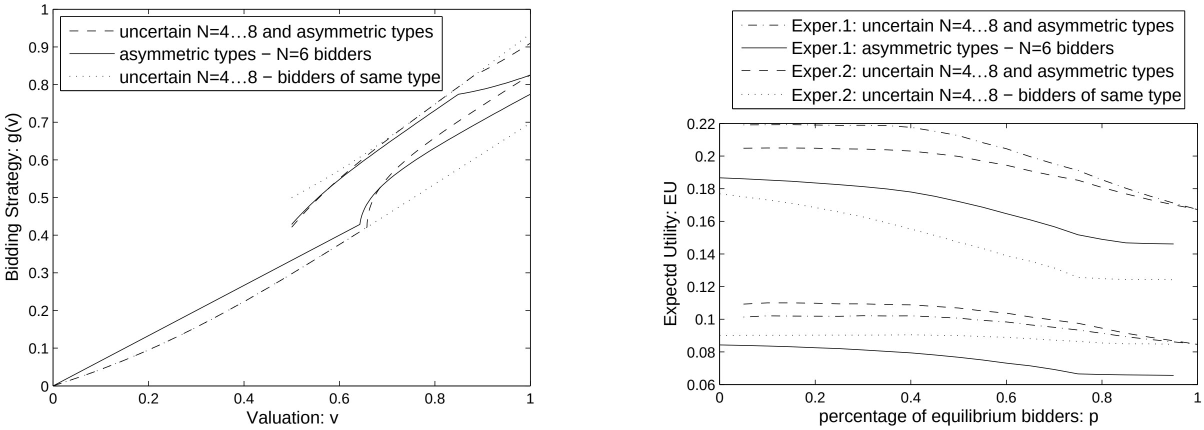


**Figure 3.** Bidding strategies used (left) and the resulting expected utilities obtained (right) in the experimental comparison of the various agents. Bidder type 1's strategies are those for valuations $v \geq 0.5$ (left) and the corresponding expected utilities for this type (1) are the top lines, where $EU > 0.12$ (right).

strategies to compare against, that is the reason for extending these state-of-the-art strategies. Therefore, we conducted two experiments, where our equilibrium analysis is compared respectively against:
(a) analysis that only considers asymmetric bidders, but assumes that there are $N = 6$ bidders participating (i.e., the mean number), and
(b) analysis that only considers that there is a stochastic number of bidders $N = 4 \ldots 8$, but ignores that they are asymmetric.

In each experiment, we vary the percentage $p$ of agents using our equilibrium strategy from $0\%$ to $100\%$ (increasing by $5\%$). We computed the expected utility of an agent using each strategy by taking $10$ million samples for each data point (equally divided for all cases: $N = 4 \ldots 8$ bidders). To keep the percentage $p$ correct, we compute the product $pN$. If this is an integer, then this is the number of bidders (out of $N$ total) who use the equilibrium strategy. If it is not, then $\lfloor pN \rfloor$ or $\lceil pN \rceil$ bidders are used in the appropriate number of samples, so that the average number of equilibrium strategy bidders is $pN$. The results of our experiments are presented in figure 3 (right). Note that because of taking 10 million samples, the error of the computation is tiny, therefore no errorbars are shown. In addition, we multiply by $4$ the expected utility of the all agents representing bidder type $2$ to make the differences easier to see. What we observe is that in general, it is always beneficial to use the equilibrium strategy derived from the analysis presented in this paper. In fact, in all cases this gives between $17\%$ and $47.3\%$ improvement (typically around $20-25\%$), with the exception of the performance for bidder type $2$ in the second experiment: for $p \leq 50\%$, the improvement is $19-21\%$, but for larger $p$ it drops sharply to only $2.6\%$ (for $p = 95\%$).

## 7 Conclusions

In this paper, we examined how an intelligent agent should bid in multi-unit auctions with a stochastic number of asymmetric bidders, where such bidders can have varying valuations and risk-attitudes. In so doing, we significantly extended the state-of-the-art, which had only examined single-unit (rather than multi-unit) auctions with either (a) a stochastic number of bidders of the same type, or (b) a known number of bidders of different types whose valuations were drawn from different priors (all with the same minimum value). In our analysis, we examined all these issues together with asymmetric risk-attitudes in a multi-unit auction case. We described the equilibrium bidding strategies in three different settings that are distinguished by the progressively greater levels of uncertainty about the types of other agents participating in the auction. Our experiments showed that using the equilibrium strategies derived from our full analysis led to significantly increased utility (usually about $20-25\%$) for the agents compared to using other strategies which are derived from (or are even better than) the previous state-of-the-art.

The main direction of future work we aim to pursue stems from our desire to apply our analysis to a wide range of real-world applications. Our motivating examples such as using computational service auctions require us to examine a stochastic number of asymmetric bidders, however it is sometimes the case that such bidders are interested in multiple services rather than a single one. We plan to take the analysis we conducted so far about how to bid for a single item, and use it to design trading agents that can participate in multiple such auctions. Optimality will probably have to be sacrificed. However, as shown in [16], optimal strategies of the same essence as those computed in this paper can be invaluable for designing trading agents for the more complex setting of purchasing multiple services (or goods).

## ACKNOWLEDGEMENTS

This research was undertaken under the ALADDIN and ORCHID projects, and completed with a grant from the Institute of Informatics and Telecommunications, NCSR Demokritos, Greece.

## REFERENCES

[1] A. Anandasivam and D. Neumann, 'Managing revenue in grids', in *42nd Hawaii Int Conf on System Sciences (HICSS '09)*, 1–10, (2009).
[2] R. Buyya, D. Abramson, and S. Venugopal, 'The grid economy', *Proceedings of the IEEE*, **93**(3), 698 –714, (2005).
[3] R. Buyya, C. Yeo, S. Venugopal, J. Broberg, and I. Brandic, 'Cloud computing and emerging it platforms: Vision, hype, and reality for delivering computing as the 5th utility', *Future Generation Computer Systems*, **25**(6), 599–616, (2009).
[4] W. Güth, R. Ivanova-Stenzel, and E. Wolfstetter, 'Bidding behavior in asymmetric auctions: An experimental study', *European Economic Review*, **49**, 1891–1913, (2005).
[5] R. Harstad, J. Kagel, and D. Levin, 'Equilibrium bid functions for auctions with an uncertain number of bidders', *Economic Letters*, **33**(1), 35–40, (1990).
[6] V. Krishna, *Auction theory*, Academic Press, 2002.
[7] B. Lebrun, 'First price auctions in the asymmetric n bidder case', *International Economic Review*, **40**(1), 125–142, (1999).
[8] Y. Liu, R. Goodwin, and S. Koenig, 'Risk-averse auction agents', in *AAMAS-03*, 353–360, (2003).
[9] E. Maskin and J. Riley, 'Asymmetric auctions', *Review of Economic Studies*, **67**(3), 413–38, (2000).
[10] R.P. McAfee and J. McMillan, 'Auctions with a stochastic number of bidders', *Journal of Economic Theory*, **43**(1), 1–19, (1987).
[11] M. Mihailescu and Y. M. Teo, 'Dynamic resource pricing on federated clouds', *Int. Conf. on Cluster, Cloud and Grid Computing (CCGRID)*, 513–517, (2010).
[12] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery, *Numerical Recipes*, Cambridge University Press, 2007.
[13] J. A. Rice, *Mathematical Statistics and Data Analysis*, Duxbury Press, 1995.
[14] S. Stein, T. R. Payne, and N. R. Jennings, 'Flexible provisioning of web service workflows', *ACM Transactions on Internet Technology*, **9**(1), 2:1–2:45, (2009).
[15] I. A. Vetsikas and N. R. Jennings, 'Outperforming the competition in multi-unit sealed bid auctions', in *AAMAS-07*, 702–709, (2007).
[16] I. A. Vetsikas, N. R. Jennings and B. Selman, 'Generating Bayes-Nash equilibria to design autonomous trading agents', in *IJCAI-07*, 1543–1550, (2007).
[17] I. A. Vetsikas and N. R. Jennings, 'Multi-unit Auctions with Asymmetric Bidders', in *IAT4EB* workshop of *ECAI-10*, 1–6, (2010).